Question Bank
Mathematics - II (BTAM203-18)

Department of Applied Sciences

1 Differential Equations

1.1 Leibnitz linear equation

1. $(x+1)\frac{dy}{dx} - y = e^x(1+x)^2$
2. $\cos^2 x\frac{dy}{dx} + y = \tan x$
3. $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$
4. $(x+2y^3)\frac{dy}{dx} = y$
5. $e^{-y}\sec^2 y dy = dx + xdy$

1.2 Bernoulli's equation

1. $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$
2. $(x^3y^2 + xy)dx = dy)$
3. $\frac{dy}{dx} + y\tan x = y^3\cos x$
4. $y - \cos x\frac{dy}{dx} = y^2(1-\sin x)\cos x$
5. $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^2 x$

1.3 Exact differential equations

1. $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$
2. $(\sec x\tan x\tan y - e^x)dx + \sec x\sec^2 y dy = 0$
3. $(y\cos x + 1)dx + \sin x dy = 0$

4. $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)dy=0$

5. $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

1.4 Equations reducible to exact equations

1. $xdy-ydx=(x^2+y^2)dx$
2. $y(2xy+e^x)dx-e^x dy=0$
3. $(3xy^2-y^3)dx-(2x^2y-xy^2)dy=0$
4. $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$
5. $(2x^2y^2+y)dx+(3x-x^3y)dy=0$

1.5 Equations of first order and higher degree

1. $yp^2+(x-y)p-x=0$
2. $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$
3. $p(p+y)=x(x+y)$
4. $16x^2+2p^2y-p^3x=0$
5. $y=x+2\tan^{-1}p$
6. $y=2px-p^2$
7. $p^3-4xyp+8y^2$
8. $y=2px+y^2p^3$
9. $p=\tan\left(x-\frac{p}{1+p^2}\right)$

1.6 Clairaut's equation

1. $(y-px)(p-1)=p$
2. $p^2(x^2-1)-2pxy+y^2-1=0$
3. $e^{3x}(p-1)+p^3e^{2y}=0$
4. $x^2(y-px)=yp^2$
5. $(y+px)^2=x^2p$

1.7 Linear differential equation of higher order

1. $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$
2. $\dfrac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \dfrac{x}{2}$
3. $(D^2 - 3D + 2)y = 2e^x \cos \dfrac{x}{2}$
4. $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$
5. $(D^2 + 3D + 2)y = e^{e^x}$

1.8 Cauchy's homogeneous equation

1. $x^2 \dfrac{d^3y}{dx^3} - 4x \dfrac{d^2y}{dx^2} + 6 \dfrac{dy}{dx} = 4$
2. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = x^2 + 2 \log x$
3. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$
4. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 12y = x^3 \log x$
5. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 20y = (x+1)^2$

1.9 Legendre's linear equation

1. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2) \dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$
2. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x) \dfrac{dy}{dx} + y = 4 \cos \log(1+x)$
3. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x) \dfrac{dy}{dx} + 16y = 8(1+2x)^2$
4. $(3+2x)^2 \dfrac{d^2y}{dx^2} - 2(3+2x) \dfrac{dy}{dx} - 12y = 6x$

1.10 Variation of parameters

1. $\dfrac{d^2y}{dx^2} + y = \sec x$
2. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

3. $\dfrac{d^2y}{dx^2} - 4y = (1+e^x)^2$

4. $\dfrac{d^2y}{dx^2} - 4y = x^2$

5. $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$

1.11 Series solution

1. $\dfrac{d^2y}{dx^2} - y = 0$

2. $y'' - xy' + x^2y = 0$

3. $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

4. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x^2+1)y = 0$

5. $2x(1-x)\dfrac{d^2y}{dx^2} + (5-7x)\dfrac{dy}{dx} - 3y = 0$

2 Complex Analysis

2.1 Limit, continuity, differentiability and C-R equations

1. If $f(z) = \dfrac{x^3y(y-\iota x)}{x^6+y^2}$, $z \neq 0$, $f(0) = 0$, prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

2. Prove that the function $f(z) = \dfrac{x^3(1+\iota - y^3(1-\iota)}{x^2+y^2}$, $z \neq 0$, $f(0) = 0$ is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

3. Determine a, b, c, d so that the function $f(z) = (x^2+axy+by^2) + \iota(cx^2+dxy+y^2)$ is analytic.

4. Show that the polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

5. Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although C-R equations are satisfied.

2.2 Analytic functions

1. Show that $u + \iota v = \dfrac{x - \iota y}{x - \iota y + a}$, $a \neq 0$ is not an analytic function whereas $u - \iota v$ is such a function.

2. Show that the function $f(z) = \dfrac{x^2 y^3 (x + \iota y)}{x^6 + y^{10}}$, $z \neq 0$, $f(0) = 0$ is not analytic at the origin even though it satisfies C-R equaitons at the origin.

3. If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \log |f'(z)| = 0$.

4. If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |Ref(z)|^2 = 2|f'(z)|^2$.

5. Prove that the function $\sinh z$ is analytic and find its derivative.

2.3 Harmonic functions

1. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic.

2. Determine the analytic function whose real part is $e^{-x}(x \sin y - y \cos y)$.

3. Find the regular function whose imaginary part is $\dfrac{x - y}{x^2 + y^2}$.

4. If $f(z) = u + \iota v$ is an analytic function, find $f(z)$ if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2 \cos x - e^y - e^{-y}}$ provided $f\left(\dfrac{\pi}{2}\right) = 0$.

5. If $f(z) = u + \iota v$ is an analytic function, find $f(z)$ if $u + v = \dfrac{x}{x^2 + y^2}$ provided $f(1) = 1$.

2.4 Conformal mappings

1. Determine the region of the w-plane into which the rectangular region in the $z-$plane bounded by the lines $x = 0, y = 0, x = 1, y = 2$ is mapped under the transformation $w = z + (2 - \iota)$.

2. Find the image of the semi-infinite strip $x > 0, 0 < y < 2$ under the transformation $w = \iota z + 1$.

3. Prove that $w = \dfrac{z}{\iota - z}$ maps the upper half of the $z-$plane into the upper half of the $w-$plane.

4. Show that under the transformation $w = \dfrac{z - \iota}{z + \iota}$, real-axis in the $z-$plane is mapped into the circle $|w| = 1$.

5. Find the image of the circle $|z - 3| = 5$ under the mapping $w = \dfrac{1}{z}$.

2.5 Bilinear transformations

1. Find the bilinear transformation that maps $1, -\iota, -1$ into the points $\iota, 0, -\iota$.
2. Find the bilinear transformation that maps $1, \iota, -1$ into the points $0, 1, \infty$.
3. Find the bilinear transformation that maps $0, -\iota, -1$ into the points $\iota, 1, \infty$.

2.6 Contour integrals

1. Evaluate $\int_{1-\iota}^{2+\iota} (2x + \iota y + 1)dz$ along the curve $x = t + 1, y = 2t^2 - 1$.
2. Evaluate $\oint_C |z|^2 dz$ around the square with the vertices at $(0, 0, (1, 0), (1, 1), (0, 1)$.
3. Prove that $\int_C \frac{1}{z} dz = -\pi\iota \ or\ \pi\iota$ according as C is the semi-circular arc $|z| = 1$ from -1 to 1 above or below the real axis.
4. Evaluate $\oint_C \log z dz$ where C is the unit circle $|z| = 1$.
5. Evaluate $\int_C (y - x - 3x^2\iota)dz$ where C is the straight line from $z = 0$ to $z = 1 + \iota$.

2.7 Cauchy integral theorem/Cauchy integral formula

1. Evaluate $\oint_C \frac{z^2 + 5}{z - 3} dz$ where C is the circle
   a) $|z| = 4$ b) $|z| = 1$
2. Evaluate $\oint_C \frac{3z^2 + 7z + 1}{z + 1} dz$ where C is the circle
   a) $|z| = 1.5$ b) $|z + \iota| = 1$
3. Evaluate $\oint_C \frac{\cos \pi z}{z^2 - 1} dz$ where C is a rectangle with vertices
   a) $2 \pm \iota, -2 \pm \iota$ b) $-\iota, 2 - \iota, 2 + \iota, \iota$
4. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z - 1)(z - 2)} dz$ where C is the circle $|z| = 3$.
5. Evaluate $\oint_C \frac{e^z}{(z + 3)(z + 2} dz$ where C is the circle $|z - 1| = \frac{1}{2}$.

2.8 Cauchy residue theorem

1. Evaluate $\oint_C \frac{z^2 + 2z - 2}{z - 4} dz$ where C is a closed curve containing $z = 4$ in its interior.
2. Evaluate $\oint_C \frac{z}{(z - 1)(z - 2)^2} dz$ where C is the circle $|z - 2| = \frac{1}{2}$.

3. Evaluate $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}dz$ where C is the circle $|z|=2$.

4. Evaluate $\oint_C \frac{z\sec z}{(1-z)^2}dz$ where C is the circle $|z|=2$.

5. Evaluate $\oint_C \tan z dz$ where C is the circle $|z|=2$.

2.9 Taylor's and Laurent's series

1. Expand $\frac{z^2-1}{(z+2)(z+3)}$ for $|z|>3$.

2. Expand $\frac{1-\cos z}{z^3}$ about $z=0$.

3. Expand $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region
   a) $|z|<1$ b) $1<|z|<4$ c) $|z|>4$

4. Expand $\sin z$ about $z=\frac{\pi}{4}$.

5. Expand $\frac{e^z}{(z-1)^2}$ about $z=1$.

2.10 Evaluation of definite integrals

1. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$, $0<a<1$.

3. Evaluate $\int_0^{\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$

4. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{1-2p\cos\theta+p^2}d\theta,\ 0<p<1$

5. Evaluate $\int_0^{\pi} \frac{a}{a^2+\sin^2\theta}d\theta$