TOPIC-MAXWELL’S EQUATIONS

-ELECTROMAGNETIC WAVES

SUBJECT-ELECTROMAGNETISM

SUBJECT CODE-BTPH103-18

OUTLINE

- Maxwell’s Equations (Integral and Differential form)
- Wave equation of an em wave in free space
- Transverse relation between E and H
- Wave equation of an em wave in conducting medium
- Reflection from a perfect conductor
- Reflection from a perfect dielectric
- Poynting vector

Maxwell's Equation's in integral form

$$\oint\!\!\!\oint_A \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_o} = \frac{1}{\varepsilon_o}\iiint_V \rho\, dV$$

Gauss's Law of electrostatics

$$\oint\!\!\!\oint_A \vec{B}\cdot d\vec{A} = 0$$

Gauss's Law for Magnetism

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_A \vec{B}\cdot d\vec{A}$$

Faraday's Law

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_o I_{encl} + \mu_o\varepsilon_o\frac{d\Phi_E}{dt} = \mu_o\iint_A\left(\vec{J} + \varepsilon_o\frac{d\vec{E}}{dt}\right)\cdot d\vec{A}$$

Ampere's Law

Maxwell's Equations

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_o}$$

Gauss's law (electricity):

- The total electric flux through any closed surface equals the net charge inside that surface divided by e_o
- This relates an electric field to the charge distribution that creates it

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

- Gauss's law (magnetism):
- The total magnetic flux through any closed surface is zero
- This says the number of field lines that enter a closed volume must equal the number that leave that volume
- This implies the magnetic field lines cannot begin or end at any point

Maxwell’s Equations

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

Faraday’s law of Induction:

- This describes the creation of an electric field by a changing magnetic flux
- The law states that the emf, which is the line integral of the electric field around any closed path, equals the rate of change of the magnetic flux through any surface bounded by that path

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I + \varepsilon_o \mu_o \frac{d\Phi_E}{dt}$$

Ampere-Maxwell law is a generalization of Ampere’s law

- It describes the creation of a magnetic field by an electric field and electric currents

Maxwell's Equation's in free space

Since there is no charge or current, the general Maxwell's equations reduce to

$$\oint\!\!\!\iint_A \vec{E} \cdot d\vec{A} = 0$$

$$\oint\!\!\!\iint_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_A \vec{B} \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o \varepsilon_o \frac{d\Phi_E}{dt} = \mu_o \varepsilon_o \frac{d}{dt}\iint_A \vec{E} \cdot d\vec{A}$$

Maxwell's equations in differential form

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

$$\nabla . \mathbf{D} = \rho$$

$$\nabla . \mathbf{B} = 0$$

Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_o \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

SI Units

- J Amp/ metre^2
- D Coulomb/metre^2
- H Amps/metre
- B Tesla

- E Volt/metre
- ε Farad/metre
- μ Henry/metre
- σ Siemen/metre

Maxwell's Equations

Name	Integral Form	Differential Form
Gauss's Law	$\oint_S \vec{E} \bullet d\vec{A} = \frac{Q}{\varepsilon_o}$	$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$
Gauss's Law	$\oint_S \vec{B} \bullet d\vec{A} = 0$	$\nabla \bullet \vec{B} = 0$
Faraday's Law	$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampere-Maxwell's Law	$\oint \vec{B} \bullet d\vec{s} = \mu_o i + \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$	$\nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$

Wave equation in free space

In free space conductivity $\sigma=0$

Since $J=\sigma E$, therefore J=0

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt} = \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

Taking curl of both sides of latter equation:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\mu_o \frac{\partial}{\partial t}\nabla \times \mathbf{H}$$

$$= -\mu_o \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Wave equation in free space (cont.....)

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

for any vector A, $\nabla \times \nabla \times \mathbf{A} = \nabla \nabla . \mathbf{A} - \nabla^2 \mathbf{A}$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator

$$\nabla \nabla . \mathbf{E} - \nabla^2 \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Since there are no free charges in free space so $\nabla . \mathbf{E} = \rho = 0$ and we get

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Transverse relation between E and H

Consider a uniform plane wave, propagating in the *z* direction

Therefore,

$$\frac{\partial \mathbf{E}}{\partial x} = 0 \qquad \frac{\partial \mathbf{E}}{\partial y} = 0$$

In a source free region, $\nabla.\mathbf{D}=\rho=0$ (Gauss' law) :

$$\nabla.\mathbf{E} = \frac{\partial \mathbf{E}_x}{\partial x} + \frac{\partial \mathbf{E}_y}{\partial y} + \frac{\partial \mathbf{E}_z}{\partial z} = 0$$

Since **E** is independent of *x* and *y*, so

$$\frac{\partial E_x}{\partial x} = 0, \quad \frac{\partial E_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow E_z = 0 \quad (E_z = \text{const} \ \text{ is not a wave})$$

- So for a plane wave, **E** has no component in the direction of propagation. Similarly for **H**.
- Plane waves have only transverse **E** and **H** components.

Phase relationship between E and H

Consider a linearly polarised wave that has a transverse component in (say) the *y* direction only:

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$E_y = E_o f(z - vt)$$

$$\Rightarrow \varepsilon \frac{\partial E_y}{\partial t} = -\varepsilon v E_o f'(z - vt) \quad = \frac{\partial H_x}{\partial z}$$

$$\Rightarrow \quad H_x = -\varepsilon v E_o \int f'(z - vt)\,\mathrm{d}z + const \quad = -\varepsilon v E_o f(z - vt)$$

$$= -\varepsilon v E_y$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}} E_y$$

Similarly

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = \mu_o \frac{\partial H_y}{\partial t}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$

Thus H and E are in phase and orthogonal

Wave equation for a conducting medium

wave equation with $J \neq 0$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\mu_o \frac{\partial}{\partial t}\nabla \times \mathbf{H}$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \frac{\partial}{\partial t}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$= -\mu_o \frac{\partial \mathbf{J}}{\partial t} - \mu_o \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

$$= -\mu_o \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Wave equation for a conducting medium (contd..)

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \sigma \frac{\partial E}{\partial t} - \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla\nabla .E - \nabla^2 E = -\mu_o \sigma \frac{\partial E}{\partial t} - \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

In the absence of sources $\nabla .\mathbf{E} = \rho = 0$

$$\nabla^2 \mathbf{E} = \mu_o \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

This is the wave equation for a decaying wave

Reflection and refraction of plane waves

- At a discontinuity the change in μ, ε and σ results in partial reflection and transmission of a wave

  For example, consider normal incidence:

$$\text{Incident wave} = E_i e^{j(\omega t - \beta z)}$$

$$\text{Reflected wave} = E_r e^{j(\omega t + \beta z)}$$

Where E_r is a complex number determined by the boundary conditions

Reflection at a perfect conductor

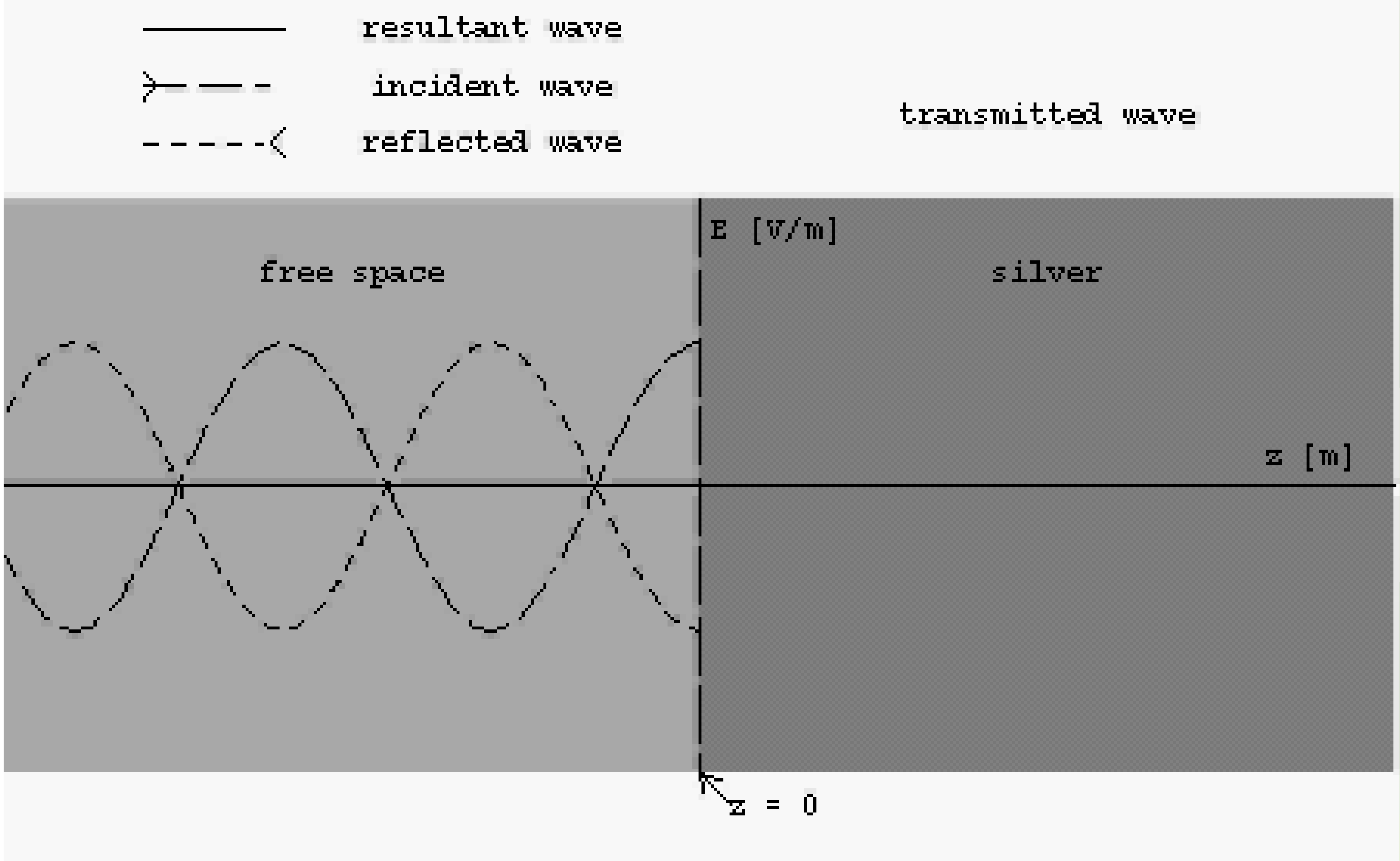
- Tangential **E** is continuous across the boundary
- For a perfect conductor **E** just inside the surface is zero
- *E* just outside the conductor must be zero

$$E_i + E_r = 0$$

$$\Rightarrow E_i = -E_r$$

- Amplitude of reflected wave is equal to amplitude of incident wave, but reversed in phase

Reflection from a perfect conductor



Reflection from a perfect conductor

- Direction of propagation is given by **E**×**H**
- If the incident wave is polarised along the *y* axis:

$$E_i = \mathbf{a}_y E_{yi}$$

$$\Rightarrow H_i = -\mathbf{a}_x H_{xi}$$

then $$\mathbf{E} \times \mathbf{H} = \left(-\mathbf{a}_y \times \mathbf{a}_x\right) E_{yi} H_{xi}$$

$$= +\mathbf{a}_z E_{yi} H_{xi}$$

That is, a z-directed wave.

For the reflected wave $\mathbf{E} \times \mathbf{H} = -\mathbf{a}_z E_{yi} H_{xi}$ and $E_r = -\mathbf{a}_y E_{yi}$

So $H_r = -\mathbf{a}_x H_{xi} = H_i$ and the magnetic field is reflected without change in phase

Reflection from a perfect conductor

Since $\cos\phi = \dfrac{e^{j\phi} + e^{-j\phi}}{2}$

$$H_T(z,t) = H_i e^{j(\omega t - \beta z)} + H_r e^{j(\omega t + \beta z)}$$

$$= H_i\left(e^{j\beta z} + e^{-j\beta z}\right)e^{j\omega t}$$

$$= 2H_i \cos\beta z\; e^{j\omega t}$$

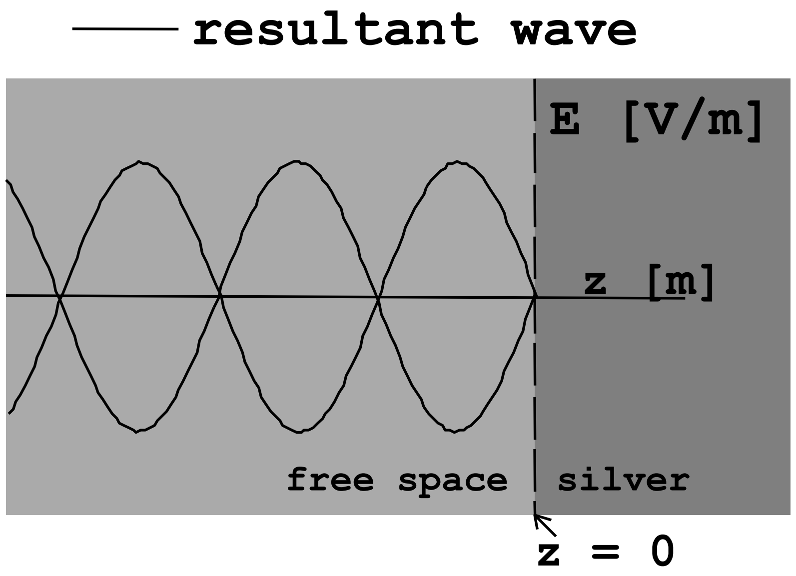
As for E_i, H_i is real (they are in phase), therefore

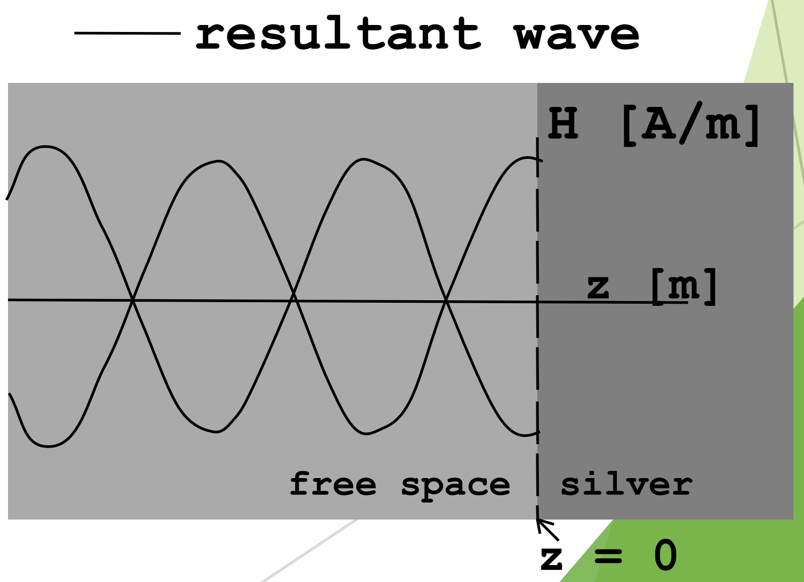
$$H_T(z,t) = \mathrm{Re}\left\{2H_i\cos\beta z\left(\cos\omega t + j\sin\omega t\right)\right\} = 2H_i\cos\beta z\cos\omega t$$

Reflection from a perfect conductor

$$H_T(z,t) = 2H_i \cos \beta z \cos \omega t$$

- Resultant magnetic field strength also has a standing-wave distribution
- In contrast to **E**, **H** has a maximum at the surface and zeros at (2*n*+1)l/4 from the surface





Reflection from a perfect conductor

$$E_T(z,t) = 2E_i \sin \beta z \sin \omega t$$

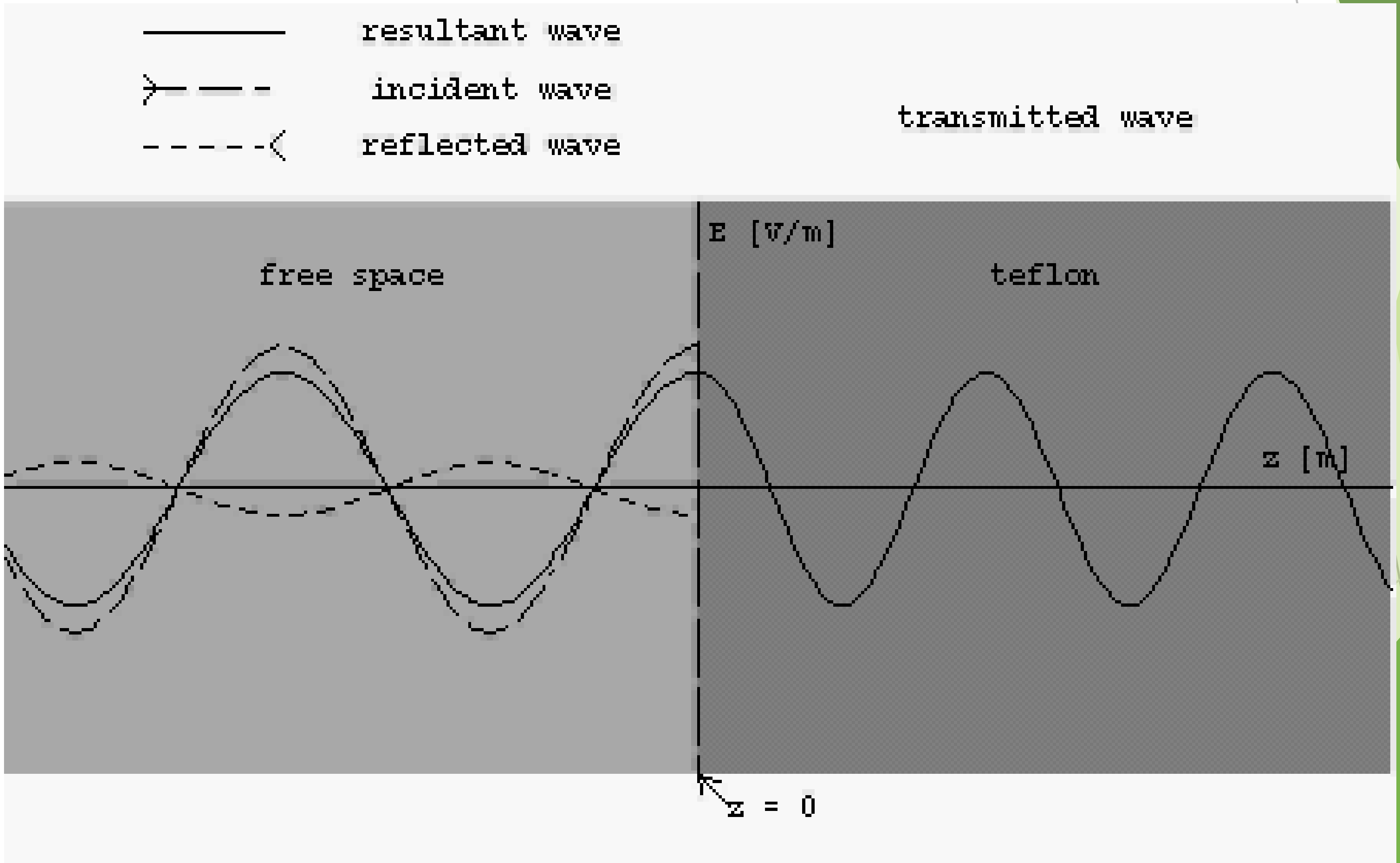
$$H_T(z,t) = 2H_i \cos \beta z \cos \omega t$$

- E_T and H_T are p/2 out of phase
- No net power flow as expected
- power flow in +z direction is equal to power flow in - z direction

Reflection by a perfect dielectric

- Reflection by a perfect dielectric (**J**=σ**E=0**)
  - no loss
- Wave is incident normally
  - **E** and **H** parallel to surface
- They are incident, reflected (in medium 1) and transmitted wave (in medium 2)

Reflection from a lossless dielectric



Reflection by a lossless dielectric

$$E_i = \eta_1 H_i$$

$$E_r = -\eta_1 H_r$$

$$E_t = \eta_2 H_t$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon_o\varepsilon_r}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Continuity of E and H at boundary requires:

$$E_i + E_r = E_t$$

$$H_i + H_r = H_t$$

Which can be combined to give

$$H_i + H_r = \frac{1}{\eta_1}(E_i - E_r) = H_t = \frac{1}{\eta_2}E_t = \frac{1}{\eta_2}(E_i + E_r)$$

$$\frac{1}{\eta_1}(E_i - E_r) = \frac{1}{\eta_2}(E_i + E_r)$$

$$\Rightarrow \eta_2(E_i - E_r) = \eta_1(E_i + E_r)$$

$$\Rightarrow E_i(\eta_2 - \eta_1) = E_r(\eta_2 + \eta_1)$$

$$\Rightarrow \qquad \rho_E = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Reflection by a lossless dielectric

$$E_i + E_r = E_t$$

$$H_i + H_r = H_t$$

Similarly

$$\tau_E = \frac{E_t}{E_i} = \frac{E_r + E_i}{E_i} = \frac{E_r}{E_i} + 1 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} + \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\tau_E = \frac{2\eta_2}{\eta_2 + \eta_1}$$

The transmission coefficient

Reflection by a lossless dielectric

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = \rho_H$$

$$\frac{H_t}{H_i} = \frac{\eta_1 E_t}{\eta_2 E_i} = \frac{\eta_1}{\eta_2}\frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\eta_1}{\eta_2 + \eta_1}\tau_H$$

$$\rho_E = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} = \frac{n_1 - n_2}{n_1 + n_2} = -\rho_H$$

$$\tau_E = \frac{E_r}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} = \frac{2n_1}{n_1 + n_2}$$

$$\tau_H = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} = \frac{2n_2}{n_1 + n_2}$$

Energy in electromagnetic waves

The electric and magnetic energy stored is given as

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

$$\frac{E_o}{B_o} = \frac{\omega}{k} = c = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$$

$$\left\{ \begin{aligned} u &= \varepsilon_0 E^2 \\ u &= \frac{1}{\mu_0} B^2 \\ u &= \sqrt{\frac{\varepsilon_0}{\mu_0}} E B \end{aligned} \right.$$

Poynting Vector

- Poynting vector points in the direction the wave moves
- Poynting vector gives the energy passing through a unit area in 1 sec.
- Units are Watts/m^2

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \longrightarrow S = \frac{EB}{\mu_o} = \frac{E^2}{\mu_o c} = \frac{cB^2}{\mu_o}$$

$$\left|\vec{S}\right| = cu$$