Fourier Series

7.1 General Properties

Fourier series

A Fourier series may be defined as an expansion of a function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad (7.1)$$

The coefficients are related to the periodic function f(x) by definite integrals: Eq.(7.11) and (7.12) to be mentioned later on.

The Dirichlet conditions:

(1) f(x) is a periodic function;

(2) f(x) has only a finite number of finite discontinuities;

(3) f(x) has only a finite number of extrem values, maxima and minima in the interval [0,2π].

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Express *cos nx* and *sin nx* in exponential form, we may rewrite Eq.(7.1) as

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right), \qquad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (7.2)$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n),$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad n > 0, \qquad (7.3)$$

and

$$c_0 = \frac{1}{2}a_0.$$

Completeness

One way to show the completeness of the Fourier series is to transform the trigonometric Fourier series into exponential form and compare It with a Laurent series.

If we expand f(z) in a Laurent series(assuming f(z) is analytic),

$$f(z) = \sum_{n=-\infty}^{\infty} d_n z^n. \qquad (7.4)$$

On the unit circle $z = e^{i\theta}$ and

$$f(z) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}. \qquad (7.5)$$

The Laurent expansion on the unit circle has the same form as the complex Fourier series, which shows the equivalence between the two expansions. Since the Laurent series has the property of completeness, the Fourier series form a complete set. There is a significant limitation here. Laurent series cannot handle discontinuities such as a square wave or the sawtooth wave.

We can easily check the orthogonal relation for different values of the eigenvalue n by choosing the interval $[0,2\pi]$

$$\int_0^{2\pi} \sin mx \sin nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 0, & m = 0, \end{cases} \tag{7.7}$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases} \tag{7.8}$$

$$\int_0^{2\pi} \sin mx \cos nx dx = 0 \qquad \text{for all integer m and n.} \tag{7.9}$$

By use of these orthogonality, we are able to obtain the coefficients

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

multipling $\cos mx$, and then integral from 0 to 2π

$$\int_0^{2\pi} \cos(mx) f(x) dx = \frac{a_0}{2} \int_0^{2\pi} \cos(mx) dx + \sum_{n=1}^{\infty} (a_n \int_0^{2\pi} \cos(nx)\cos(mx) dx + b_n \int_0^{2\pi} \sin(nx)\cos(mx) dx)$$

Similarly

$$\int_0^{2\pi} \sin(mx) f(x) dx = \frac{a_0}{2} \int_0^{2\pi} \sin(mx) dx + \sum_{n=1}^{\infty} (a_n \int_0^{2\pi} \cos(nx)\sin(mx) dx + b_n \int_0^{2\pi} \sin(nx)\sin(mx) dx)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt, \qquad (7.11)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt, \quad n = 0,1,2 \qquad (7.12)$$

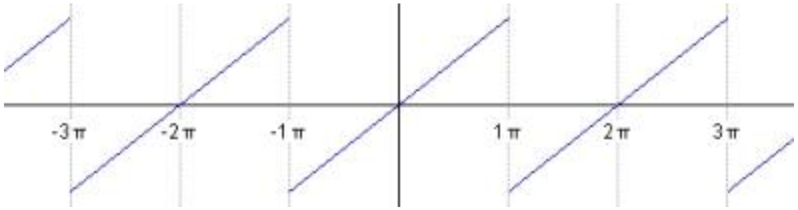
Substituting them into Eq.(7.1), we write

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi} f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos nx\int_0^{2\pi} f(t)\cos ntdt+\sin nx\int_0^{2\pi} f(t)\sin ntdt\right)$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(t)\cos n(t-x)dt, \qquad (7.13)$$

This equation offers one approach to the development of the Fourier integral and Fourier transforms.

Sawtooth wave



Let us consider a sawtooth wave

$$f(x)=\begin{cases} x, \; x & 0\le x\le\pi \\ -2\pi, & \pi\le x\le 2\pi. \end{cases} \qquad (7.14)$$

For convenience, we shall shift our interval from $[0,2\pi]$ to $[-\pi,\pi]$. In this interval

we have simply f(x)=x. Using Eqs.(7.11) and (7.12), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\sin nt\,dt$$

$$= \frac{2}{\pi n}\left[-t\cos nt\,\Big|_0^{\pi} + \int_0^{\pi}\cos nt\,dt\right]$$

$$= \frac{2}{n}(-1)^{n+1},$$

So, the expansion of f(x) reads

$$f(x) = x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots\right] \qquad (7.15)$$

Figure 7.1 shows *f(x)* for the sum of 4, 6, and 10 terms of the series. Three features deserve comment.

1.There is a steady increase in the accuracy of the representation as the number of terms included is increased.

2.All the curves pass through the midpoint $f(x) = 0$ at $x = \pi$

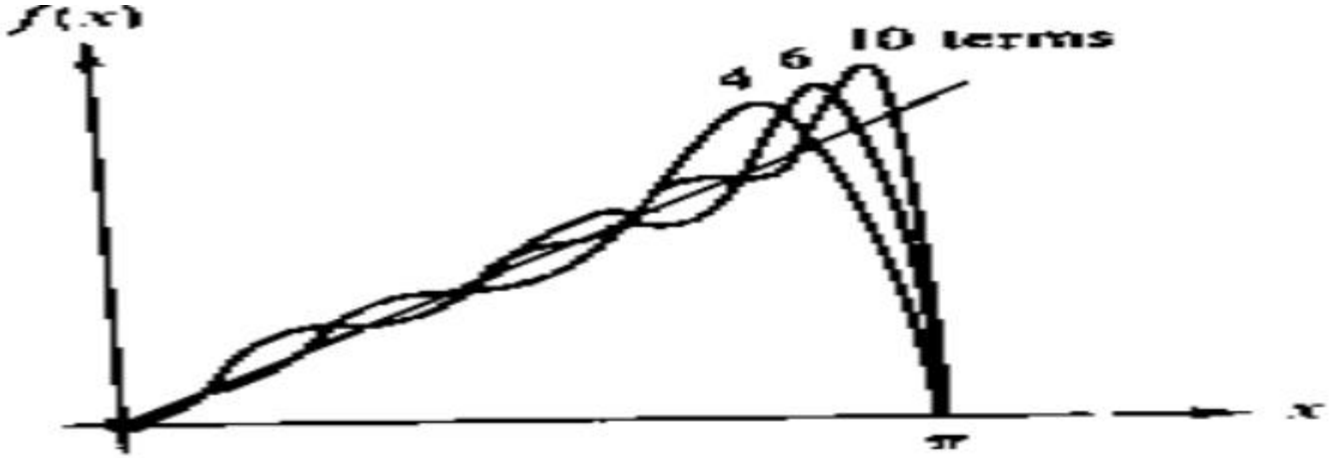
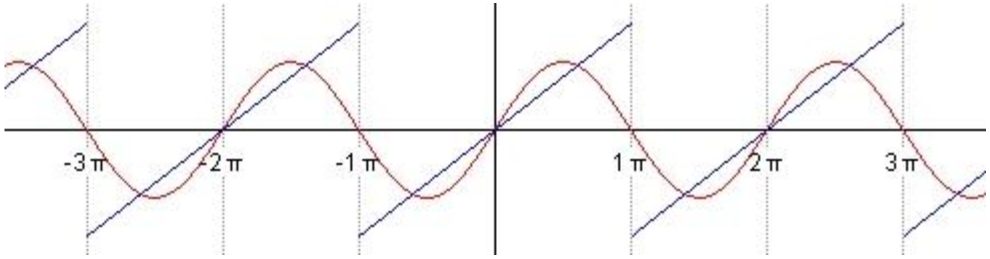


Figure 7.1 Fourier representation of sawtooth wave



- Summation of Fourier Series

Usually in this chapter we shall be concerned with finding the coefficients of the Fourier expansion of a known function. Occasionally, we may wish to reverse this process and determine the function represented by a given Fourier series.

Consider the series $\sum_{n=1}^{\infty}(1/n)\cos nx$, $x\in(0,2\pi)$. Since the series is only conditionally convergent (and diverges at x=0), we take

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n}=\lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^{n}\cos nx}{n}, \qquad (7.17)$$

absolutely convergent for $|r|<1$. Our procedure is to try forming power series by transforming the trigonometric function into exponential form:

$$\sum_{n=1}^{\infty}\frac{r^{n}\cos nx}{n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^{n}e^{inx}}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^{n}e^{-inx}}{n}. \qquad (7.18)$$

Now these power series may be identified as Maclaurin expansions of $-\ln(1-z)$

$z=re^{ix}$, re^{-ix} and

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = -\frac{1}{2}\left[\ln(1-re^{ix}) + \ln(1-re^{-ix})\right]$$

$$= -\ln\left[(1+r^2) - 2r\cos x\right]^{1/2}. \qquad (7.19)$$

Letting r=1,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln(2-2\cos x)^{1/2}$$

$$= -\ln(2\sin\frac{x}{2}), \qquad x \in (0,2\pi) \qquad (7.20)$$

Both sides of this expansion diverge as $x \to 0$ and 2π

7.2 ADVANTAGES, USES OF FOURIER SERIES

- Discontinuous Function

One of the advantages of a Fourier representation over some other representation, such as a Taylor series, is that it may represent a discontinuous function. An example id the sawtooth wave in the preceding section. Other examples are considered in Section 7.3 and in the exercises.

- Periodic Functions

Related to this advantage is the usefulness of a Fourier series representing a periodic functions . If *f(x)* has a period of 2π, perhaps it is only natural that we expand it in a series of functions with period 2π, $2\pi/2$, $2\pi/3$ $\ldots$ This guarantees that if our periodic f(x) is represented over one interval $[0,2\pi]$ or $[-\pi,\pi]$ the representation holds for all finite *x*.

At this point we may conveniently consider the properties of symmetry. Using the interval $[-\pi,\pi]$, $\sin x$ is odd and $\cos x$ is an even function of *x*. Hence , by Eqs. (7.11) and (7.12), if *f(x)* is odd, all $a_n = 0$ if *f(x)* is even all $b_n = 0$. In other words,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad f(x) \quad \text{enen,} \qquad (7.21)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad f(x) \quad \text{odd.} \qquad (7.21)$$

Frequently these properties are helpful in expanding a given function.

We have noted that the Fourier series periodic. This is important in considering whether Eq. (7.1) holds outside the initial interval. Suppose we are given only that

$$f(x) = x, \qquad 0 \le x < \pi \qquad (7.23)$$

and are asked to represent *f(x)* by a series expansion. Let us take three of the infinite number of possible expansions.

1.If we assume a Taylor expansion, we have

$$f(x) = x, \tag{7.24}$$

a one-term series. This (one-term) series is defined for all finite x.

2.Using the Fourier cosine series (Eq. (7.21)) we predict that

$$\begin{aligned} f(x) &= -x, & -\pi < x \le 0, \\ f(x) &= 2\pi - x, & \pi < x \le 2\pi. \end{aligned} \tag{7.25}$$

3.Finally, from the Fourier sine series (Eq. (7.22)), we have

$$\begin{aligned} f(x) &= x, & -\pi < x \le 0, \\ f(x) &= x - 2\pi, & \pi < x \le 2\pi. \end{aligned} \tag{7.26}$$

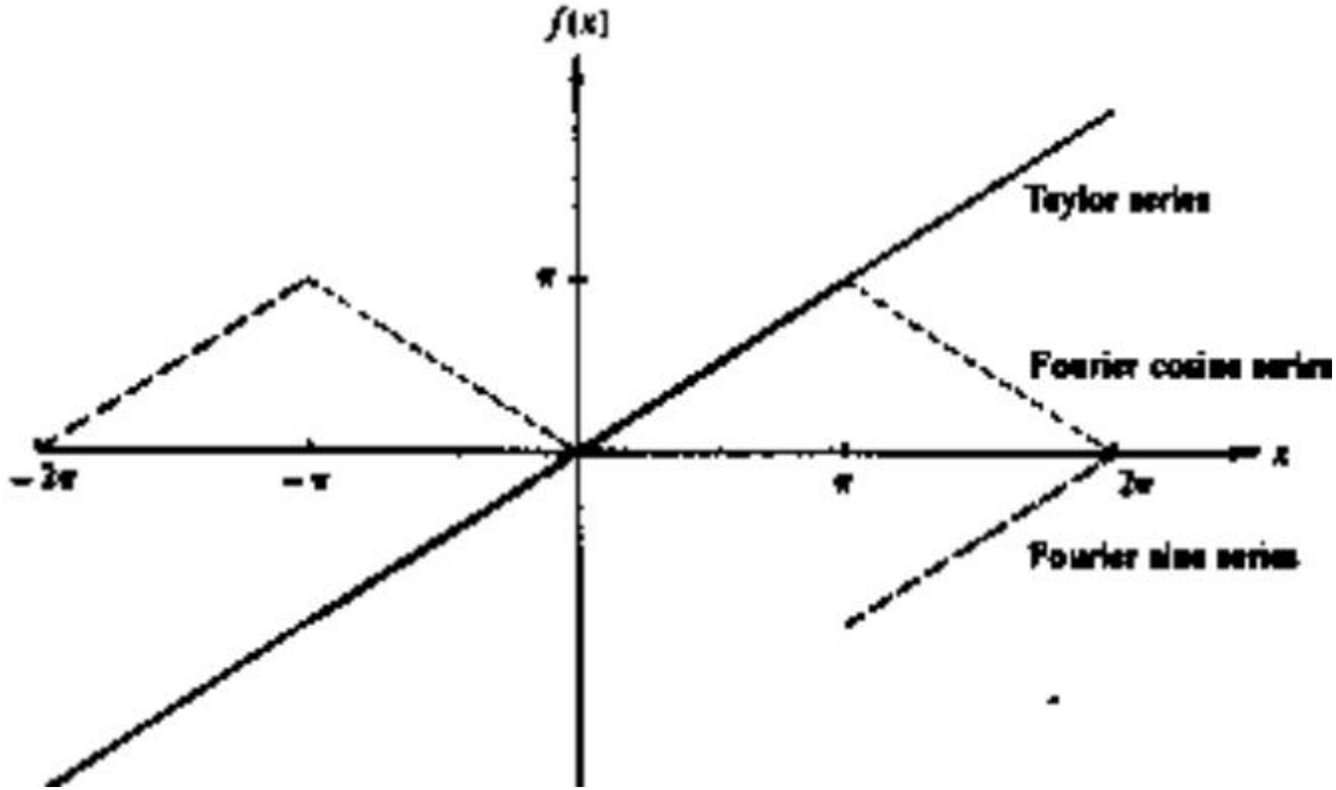


Figure 7.2 Comparison of Fourier cosine series, Fourier sine series and Taylor series.

These three possibilities, Taylor series, Fouries cosine series, and Fourier sine series, are each perfectly valid in the original interval $[0,\pi]$ Outside, however, their behavior is strikingly different (compare Fig. 7.3). Which of the three, then, is correct? This question has no answer, unless we are given more information about *f(x)*. It may be any of the three ot none of them. Our Fourier expansions are valid over the basic interval. Unless the function *f(x)* is known to be periodic with a period equal to our basic interval, or $(1/n)$ th of our basic interval, there is no assurance whatever that representation (Eq. (7.1)) will have any meaning outside the basic interval.

It should be noted that the set of functions $\cos nx$, $n = 0,1,2\ldots$, forms a complete orthogonal set over $[0,\pi]$. Similarly, the set of functions $\sin nx$, $n = 1,2,3\ldots$ forms a complete orthogonal set over the same interval. Unless forced by boundary conditions or a symmetry restriction, the choice of which set to use is arbitrary.

Change of interval

So far attention has been restricted to an interval of length of 2π. This restriction may easily be relaxed. If *f(x)* is periodic with a period $2L$, we may write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right), \tag{7.27}$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt, \qquad n = 0,1,2,3\ldots, \tag{7.28}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt, \qquad n = 0,1,2,3\ldots, \tag{7.29}$$

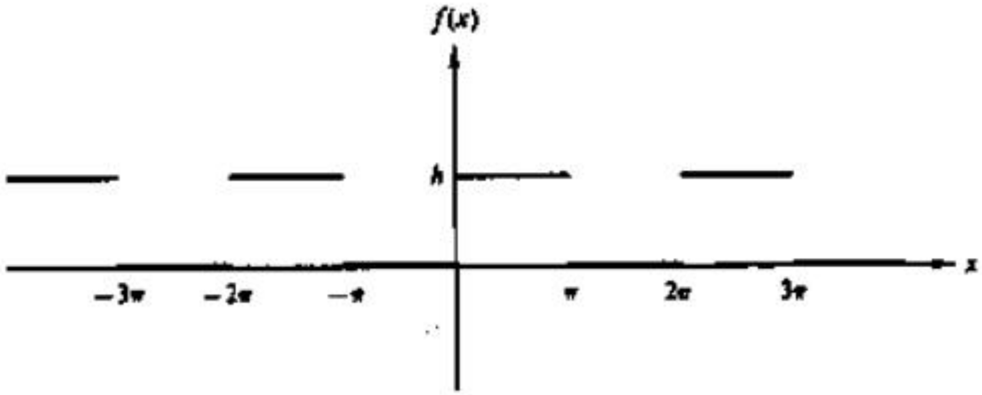
replacing x in Eq. (7.1) with $\pi x/L$ and t in Eq. (7.11) and (7.12) with $\pi t/L$
(For convenience the interval in Eqs. (7.11) and (7.12) is shifted to $-\pi \le t \le \pi$.)
The choice of the symmetric interval (-*L*, *L*) is not essential. For f(x) periodic with a period of *2L*, any interval $(x_0, x_0 + 2L)$ will do. The choice is a matter of convenience or literally personal preference.

7.3 APPLICATION OF FOURIER SERIES

Example 7.3.1 Square Wave ——High Frequency

One simple application of Fourier series, the analysis of a "square" wave (Fig. (7.5)) in terms of its Fourier components, may occur in electronic circuits designed to handle sharply rising pulses. Suppose that our wave is designed by

$$
\begin{aligned}
f(x) &= 0, && -\pi < x < 0, \\
f(x) &= h, && 0 < x < \pi.
\end{aligned}
\qquad (7.30)
$$



From Eqs. (7.11) and (7.12) we find

$$a_0 = \frac{1}{\pi}\int_0^{\pi} h\,dt = h, \qquad (7.31)$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} h\cos nt\,dt = 0 \qquad (7.32)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} h\sin nt\,dt = \frac{h}{n\pi}(1-\cos n\pi) = \begin{cases} \dfrac{2h}{n\pi}, & n = odd \\ 0, & n = even. \end{cases}; \qquad (7.33)$$

The resulting series is

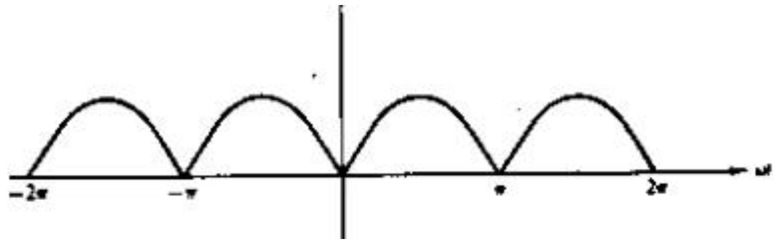
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \qquad (7.36)$$

Except for the first term which represents an average of *f(x)* over the interval $[-\pi,\pi]$ all the cosine terms have vanished. Since $f(x) - h/2$ is odd, we have a Fourier sine series. Although only the odd terms in the sine series occur, they fall only as n^{-1} This is similar to the convergence (or lack of convergence) of harmonic series. Physically this means that our square wave contains a lot of high-frequency components. If the electronic apparatus will not pass these components, our square wave input will emerge more or less rounded off, perhaps as an amorphous blob.

Example 7.3.2 Full Wave Rectifier

As a second example, let us ask how the output of a full wave rectifier approaches pure direct current (Fig. 7.6). Our rectifier may be thought of as having passed the positive peaks of an incoming sine and inverting the negative peaks. This yields

$$f(x) = \sin\omega t, \qquad 0 < \omega t < \pi,$$
$$f(x) = -\sin\omega t, \qquad -\pi < \omega t < 0. \qquad (7.37)$$



Since *f(t)* defined here is even, no terms of the form $\sin n\omega t$ will appear. Again, from Eqs. (7.11) and (7.12), we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -\sin\omega t\, d(\omega t) + \frac{1}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t)$$
$$= \frac{2}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t) = \frac{4}{\pi}, \qquad (7.38)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin\omega t \cos n\omega t\, d(\omega t)$$

$$= -\frac{2}{\pi}\frac{2}{n^2-1}, \qquad n = even \qquad (7.39)$$

$$= 0 \qquad n = odd.$$

Note carefully that $[0,\pi]$ is not an orthogonality interval for both sines and cosines together and we do not get zero for even *n*. The resulting series is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos n\omega t}{n^2-1}. \qquad (7.40)$$

The original frequency ω has been eliminated. The lowest frequency oscillation is 2ω The high-frequency components fall off as n^{-2}, showing that the full wave rectifier does a fairly good job of approximating direct current. Whether this good approximation is adequate depends on the particular application. If the remaining ac components are objectionable, they may be further suppressed by appropriate filter circuits.

These two examples bring out two features characteristic of Fourier expansion.

1. If *f(x)* has discontinuities (as in the square wave in Example 7.3.1), we can expect the *n*th coefficient to be decreasing as $1/n$. Convergence is relatively slow.

2. If *f(x)* is continuous (although possibly with discontinuous derivatives as in the Full wave rectifier of example 7.3.2), we can expect the *n*th coefficient to be decreasing as $1/n^2$

Example 7.3.3 Infinite Series, Riemann Zeta Function

As a final example, we consider the purely mathematical problem of expanding x^2. Let

$$f(x) = x^2, \qquad -\pi < x < \pi \qquad (7.41)$$

by symmetry all $b_n = 0$ For the a_n 's we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}, \tag{7.42}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx \\ &= \frac{2}{\pi}(-1)^n \frac{2\pi}{n^2} \\ &= (-1)^n \frac{4}{n^2}. \end{aligned} \tag{7.43}$$

From this we obtain

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}. \tag{7.44}$$

As it stands, Eq. (7.44) is of no particular importance, but if we set $x = \pi$

$$\cos n\pi = (-1)^n \qquad (7.45)$$

and Eq. (7.44) becomes

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (7.46)$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \equiv \xi(2), \qquad (7.47)$$

thus yielding the Riemann zeta function, $\xi(2)$, in closed form. From our expansion of x^2 and expansions of other powers of x numerous other infinite series can be evaluated.

Fourier Series

1. $$\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \\ \frac{1}{2}(\pi - x), & 0 \le x < \pi \end{cases}$$

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin nx = \frac{1}{2} x, \quad -\pi < x < \pi$$

3. $$\sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x = \begin{cases} -\pi/4, & -\pi < x < 0 \\ \pi/4, & 0 < x < \pi \end{cases}$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx = -\ln\left(2\sin\left|\frac{x}{2}\right|\right), \quad -\pi < x < \pi$$

5. $$\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} \cos nx = -\ln\left(2\cos\left(\frac{x}{2}\right)\right), \quad -\pi < x < \pi$$

6. $$\sum_{n=0}^{\infty} \frac{1}{2n+1} \cos(2n+1)x = \frac{1}{2} \ln\left(\cot\left|\frac{x}{2}\right|\right), \quad -\pi < x < \pi$$

7.4 Properties of Fourier Series

Convergence

It might be noted, first that our Fourier series should not be expected to be uniformly convergent if it represents a discontinuous function. A uniformly convergent series of continuous function ($\sin nx$, $\cos nx$) always yields a continuous function. If, however,

(a) $f(x)$ is continuous, $-\pi \leq x \leq \pi$

(b) $f(-\pi) = f(\pi)$

(c) $f'(x)$ is sectionally continuous,

the Fourier series for f(x) will converge uniformly. These restrictions do not demand that f(x) be periodic, but they will satisfied by continuous, differentiable, periodic function (period of 2π

Integration

Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{7.60}$$

yields

$$\int_{x_0}^{x} f(x)dx = \left.\frac{a_0 x}{2}\right|_{x_0}^{x} + \sum_{n=1}^{\infty} \left.\frac{a_n}{n} \sin nx\right|_{x_0}^{x} - \sum_{n=1}^{\infty} \left.\frac{b_n}{n} \cos nx\right|_{x_0}^{x}. \tag{7.61}$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series (Eq. (7.60)) is not itself convergent! The function f(x) need only be integrable.

Strictly speaking, Eq. (7.61) may be a Fourier series; that is , if $a_0 \neq 0$ there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x) - \tfrac{1}{2}a_0 x \qquad (7.62)$$

will still be a Fourier series.

Differentiation

The situation regarding differentiation is quite different from that of integration. Here thee word is *caution*. Consider the series for

$$f(x) = x, \qquad -\pi < x < \pi \qquad (7.63)$$

We readily find that the Fourier series is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \quad -\pi < x < \pi \qquad (7.64)$$

Differentiating term by term, we obtain

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx, \qquad (7.65)$$

which is not convergent ! Warning. Check your derivative

For a triangular wave which the convergence is more rapid (and uniform)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,odd}^{\infty} \frac{\cos nx}{n^2}. \qquad (7.66)$$

Differentiating term by term

$$f'(x) = \frac{4}{\pi} \sum_{n=1,odd}^{\infty} \frac{\sin nx}{n}. \qquad (7.67)$$

which is the Fourier expansion of a square wave

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases} \tag{7.68}$$

As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.

In general, term-by-term differentiation is permissible under the same conditions listed for uniform convergence.