1- Introduction

Text Book : Chapter 1, Sections: 1.1, 1.2.

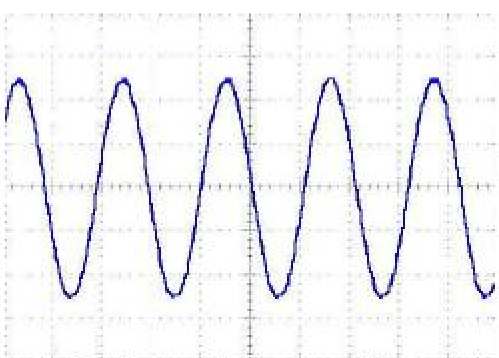
1.1 – What is Digital Signal Processing ?

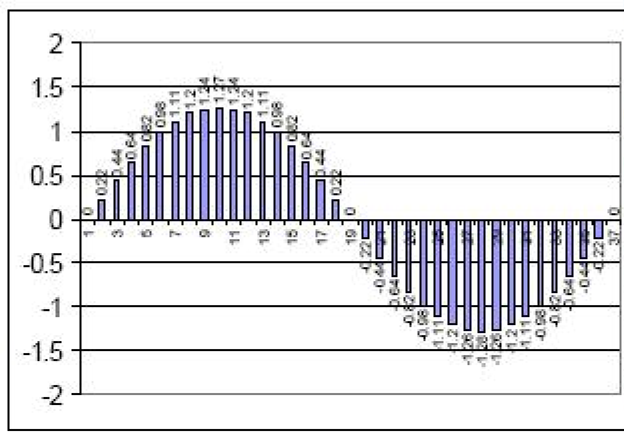
A) Digital : *Signals are either Analogue, Discrete, or Digital signals.*

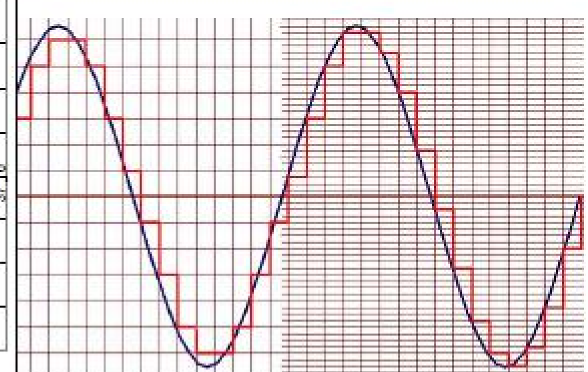
- ***Analogue Signal*** : *Continuous in both time and amplitude, any value at any time can be found.*

- ***Discrete Signal*** : *Discrete in time (sampled signal) & Continuous in amplitude.*

- ***Digital Signal*** : *Discrete in time (sampled signal) & Discrete in amplitude (Quantized Samples).*







B) __Signal__ : It is an information-bearing function, It is either:

- *1-D signal as speech.*

2-D signal as grey-scale image {i(x,y)}.

- *3-D signal as video {r(x,y,t),g(x,y,t),b(x,y,t)}.*

C) *Processing* :

*Signal Processing refers to the work of manipulating signals so that information carried can be expressed, transmitted, restored,… etc in a more **efficient** & **reliable** way by the system (hardware \ software).*

Least resource usage

Least error

- General Purpose Processors (GPP), Micro-Controllers.
- Digital Signal Processors (DSP); Dedicated Integrated Circuits. → Fast
- Programmable Logic (PLD, FPGA). → Faster

Fast, Faster → Real-time DSP'ing

- Programming Languages: Pascal, C, C++,…
- High-Level Languages: Matlab, MathCad,…
- Dedicated Tools (e.g. Filter design s/w packages).

1.2 – Why DSP ?

- *Greater Flexibility*

The same DSP hardware can be programmed and reprogrammed to perform a variety of functions.

- *Guaranteed Precision*

Accuracy is only determined by the number of bits used. (not on resistors,…etc; analogue parameters).

- *No drift in performance with temperature or age.*
- *Perfect Reproducibility*

Identical Performance from unit to unit is obtained since there are no variations due to component tolerance. e.g. a digital recording can be copied or reproduced several times with the same quality.

- *Superior Performance*

Performing tasks that are not possible with ASP, e.g. linear phase response and complex adaptive filtering algorithms.

- *DSP benefits from the tremendous advances in semiconductor technology.*

Achieving greater reliability, lower cost, smaller size, lower power consumption, and higher speed.

1.3 – DSP LIMITATIONS

- *Speed & Cost Limitations of ADC & DAC*

  Either too expensive or don't have sufficient resolution for large-bandwidth DSP applications.

- *Finite Word-Length Problems*

  Degradation in system performance may result due to the usage of a limited number of bits for economic considerations.

- *Design Time*

  DSP system design requires a knowledgeable DSP engineer possessing necessary software resources to accomplish a design in a reasonable time.

What is DSP Used For?

…And much more!

Application Areas

Image Processing
Pattern recognition
Robotic vision
Image enhancement
Facsimile
animation

Instrumentation/Control
spectrum analysis
noise reduction
data compression
position and rate control

Speech/Audio
speech recognition
speech synthesis
text to speech
digital audio
equalization

Military
secure communications
radar processing
sonar processing
missile guidance

Telecommunications
Echo cancellation
Adaptive equalization
ADPCM trans-coders
Spread spectrum
Video conferencing

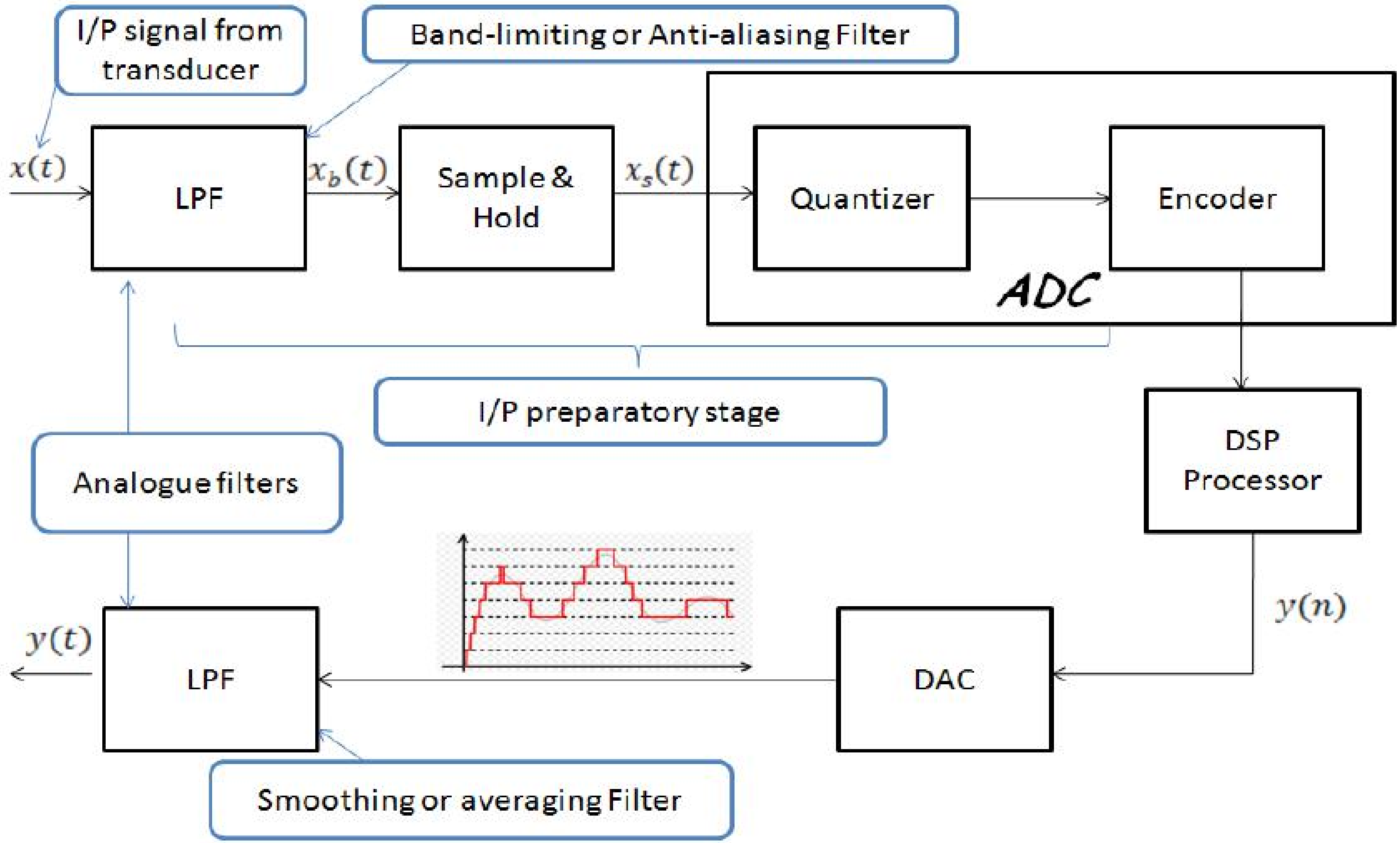
Biomedical
patient monitoring
scanners
EEG brain mappers
ECG Analysis
X-Ray storage/enhancement

Consumer applications
cellular mobile phones
UMTS (universal Mobile Telec. Sys.)
digital television
digital cameras
internet phone
etc.

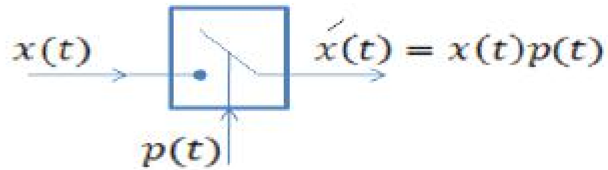
DSP Devices & Architectures

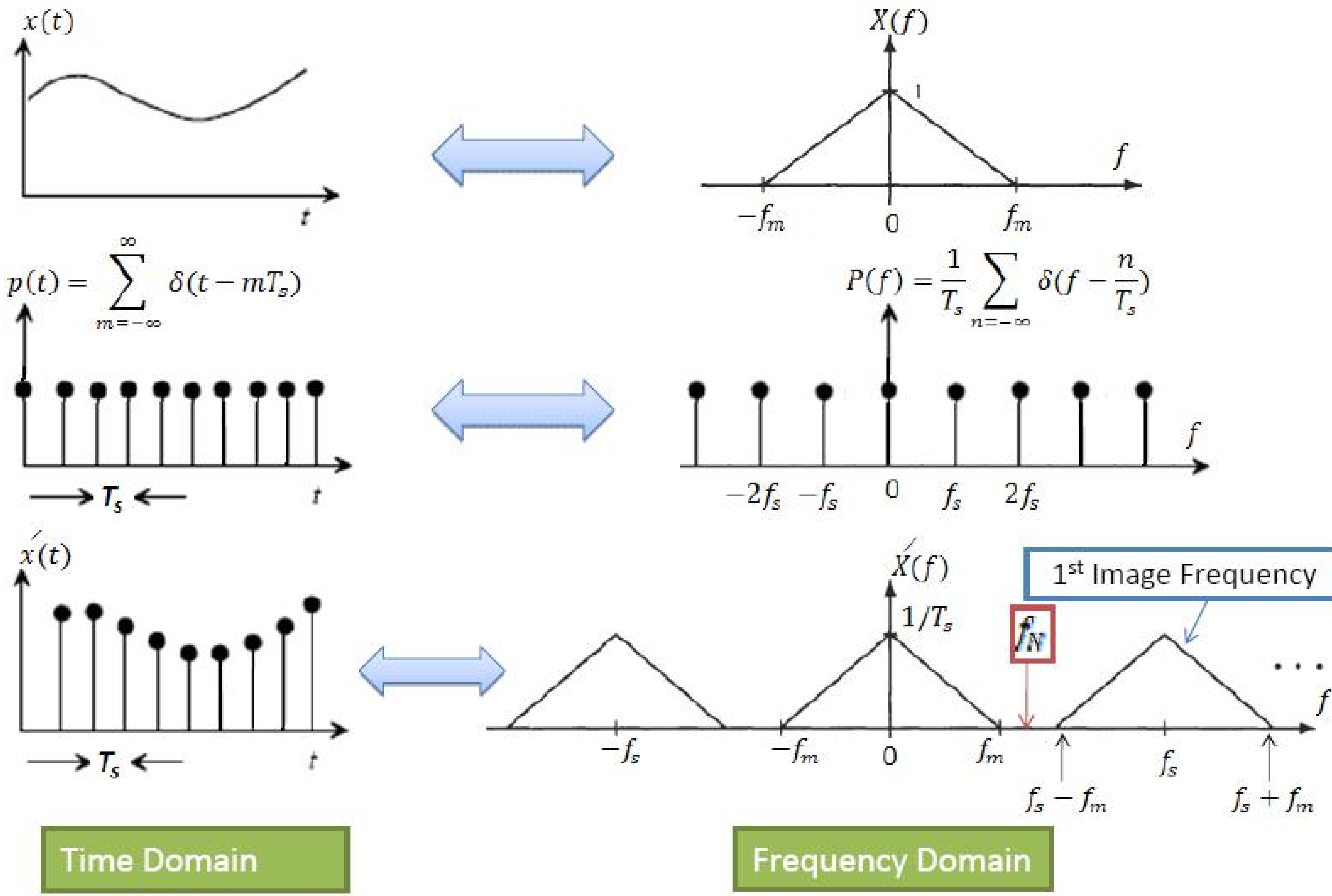
- Selecting a DSP – several choices:
  - Fixed-point;
  - Floating point;
  - Application-specific devices (e.g. FFT processors, speech recognizers,etc.).
- Main DSP Manufacturers:
  - Texas Instruments (http://www.ti.com)
  - Motorola (http://www.motorola.com)
  - Analog Devices (http://www.analog.com)

2.1 – Typical Real-Time DSP System



2.2 – Sampling Theorem & Aliasing



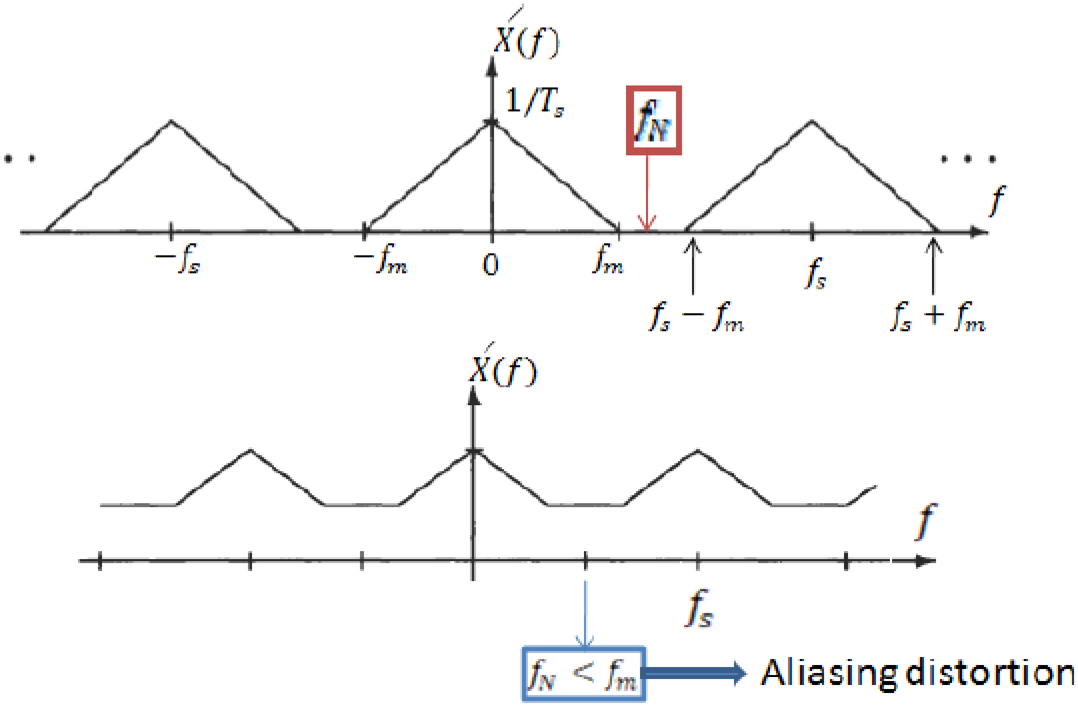


2.2 – Sampling Theorem & Aliasing - continued

Sampling Theory: $f_s \geq 2f_m$

Nyquist Frequency: $f_N = \frac{f_s}{2} \geq f_m$

so, when $\frac{f_s}{2} > f_m$ → *No aliasing*



- *In practice, aliasing is always present because of noise & the existence of signal outside the band of interest.*
- *The problem then is <u>deciding the level of aliasing that is acceptable</u> and then designing a suitable anti-aliasing filter & <u>choosing an appropriate sampling frequency</u> to achieve this.*

2.3 – Anti-aliasing Filtering

<u>To reduce the effect of aliasing:</u>

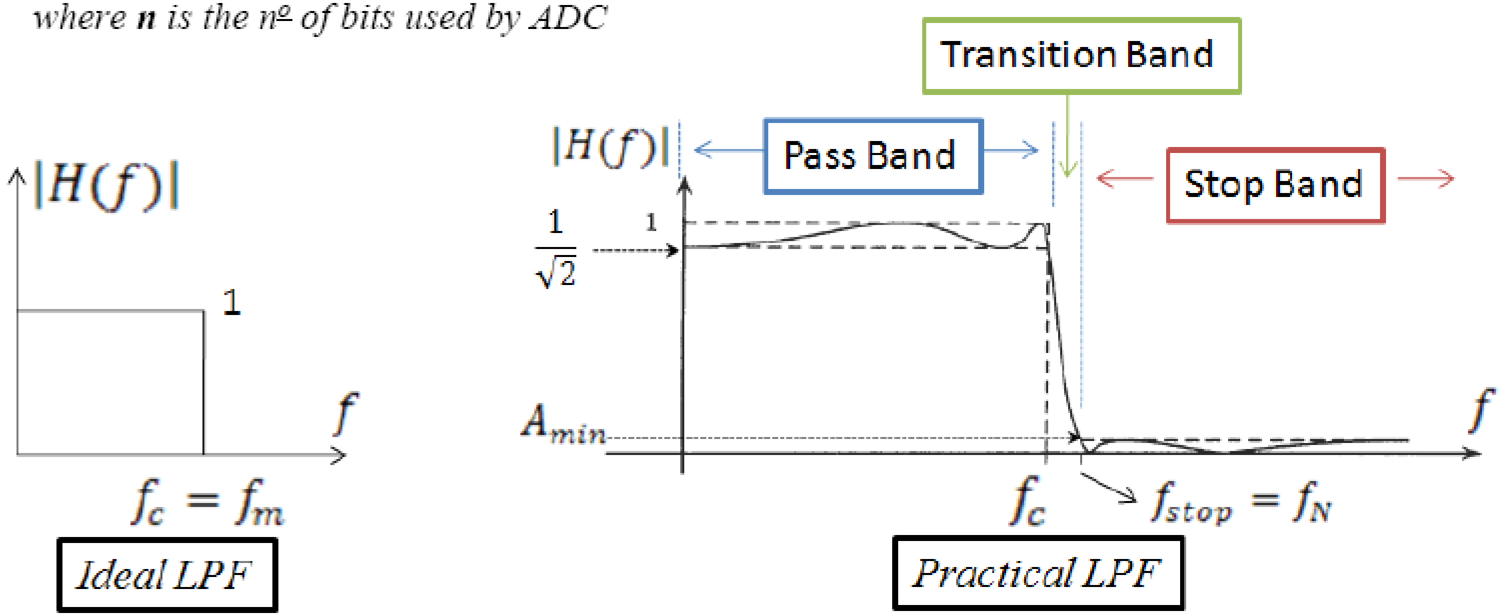
a)Sharp Cut-off anti-aliasing filters are normally used to band-limit the signal.

b)Increasing the sampling frequency to widen the separation between the signal & the image spectra.

c)Practical LPF provides sufficient attenuation at $f > f_N$; $f > f_{stop}$ *to a level not detectable be ADC,*

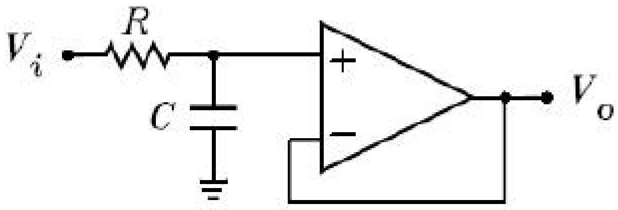
$$A_{\min} = 20\log(\sqrt{1.5} \times 2^n)$$
$$= 6.02n + 1.76 \qquad dB$$

*where **n** is the nº of bits used by ADC*



2.3.1 – Butterworth(LPF)

$$H(f) = \frac{v_o}{v_i} = \frac{1/_{j\omega c}}{R + 1/_{j\omega c}} = \frac{1}{1 + j2\pi fRC}$$



First Order LPF (N=1)

$$|H(f)| = \frac{1}{\sqrt{2}}, \text{ at } f = f_c$$

$$then,\ f_c = \frac{1}{2\pi RC}$$

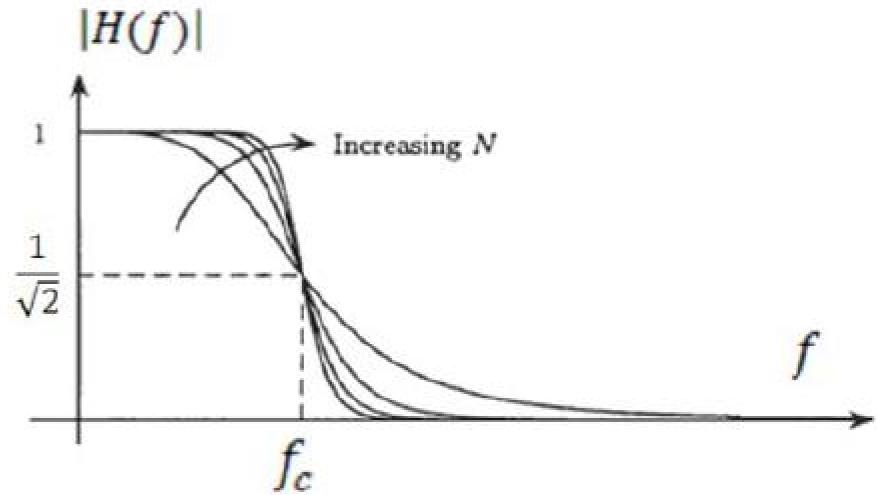
$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}}$$

$$Generally,\ |H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2N}}}$$ *; N: order of the filter*

2.3.1 – Butterworth(LPF) - continued

Higher N

- *narrower transition width (steeper roll-off).*
- *more phase distortion.*
- *allows the use of low sampling rate.*
- *slower, cheaper ADC*



Higher f_s

- *fast, expensive ADC. (real-time signal processing trend).*
- *usage of a simple anti-aliasing filter which minimizes phase distortion.*
- *Improved SNR.*

SIGNALS & SYSTEMS

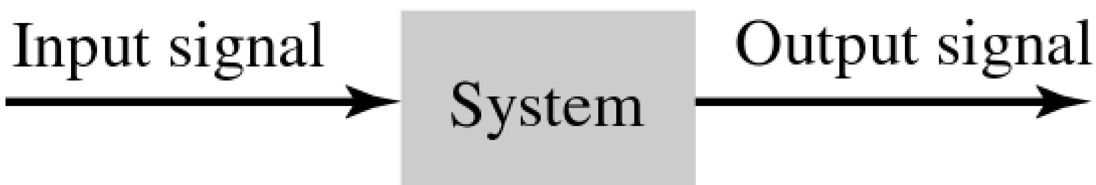
Introduction to signals & systems

Contents

- **Introduction to signals & systems**
  - Introduction to various signals & systems.
    - Signal classification
    - Useful signal model
    - Operation on signal
    - Properties of system
  - Time and frequency domains.

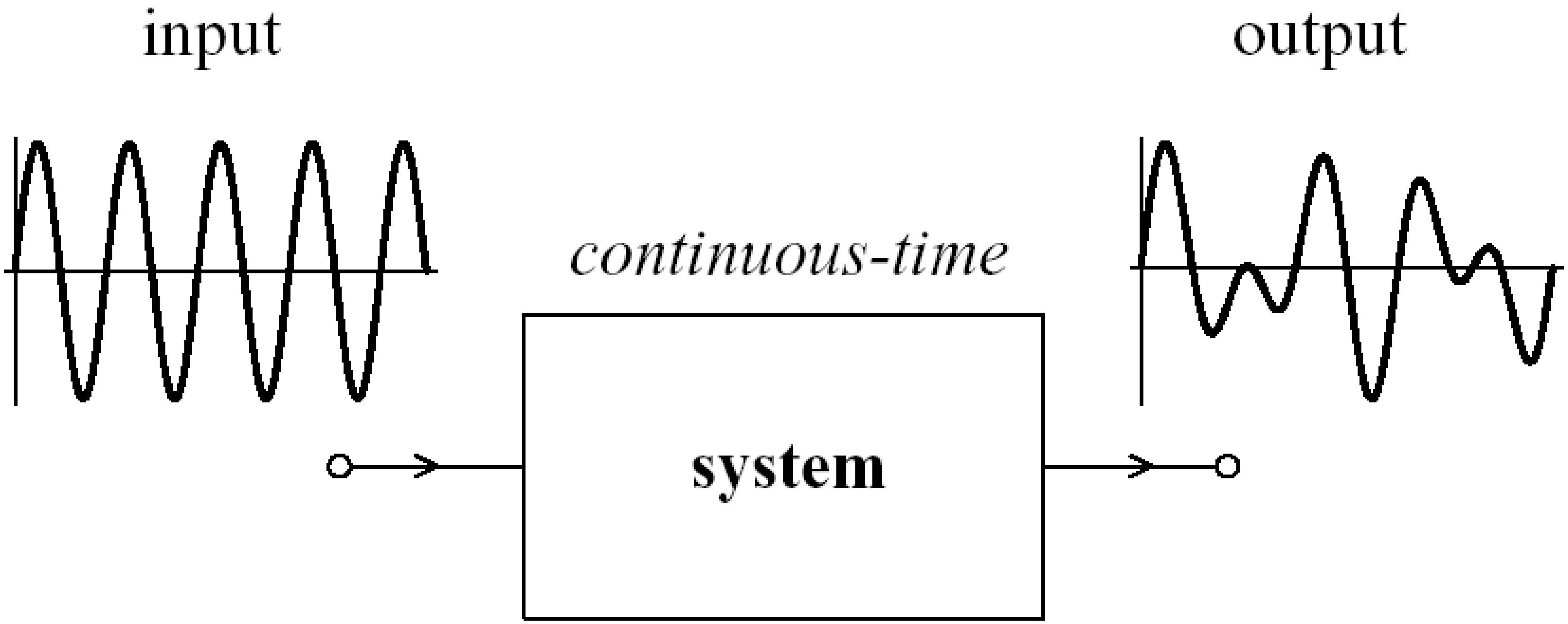
Signals & Systems

- **Signal**: a function of one or more variables that conveys information on the nature of a physical phenomenon.
- **System**: an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.



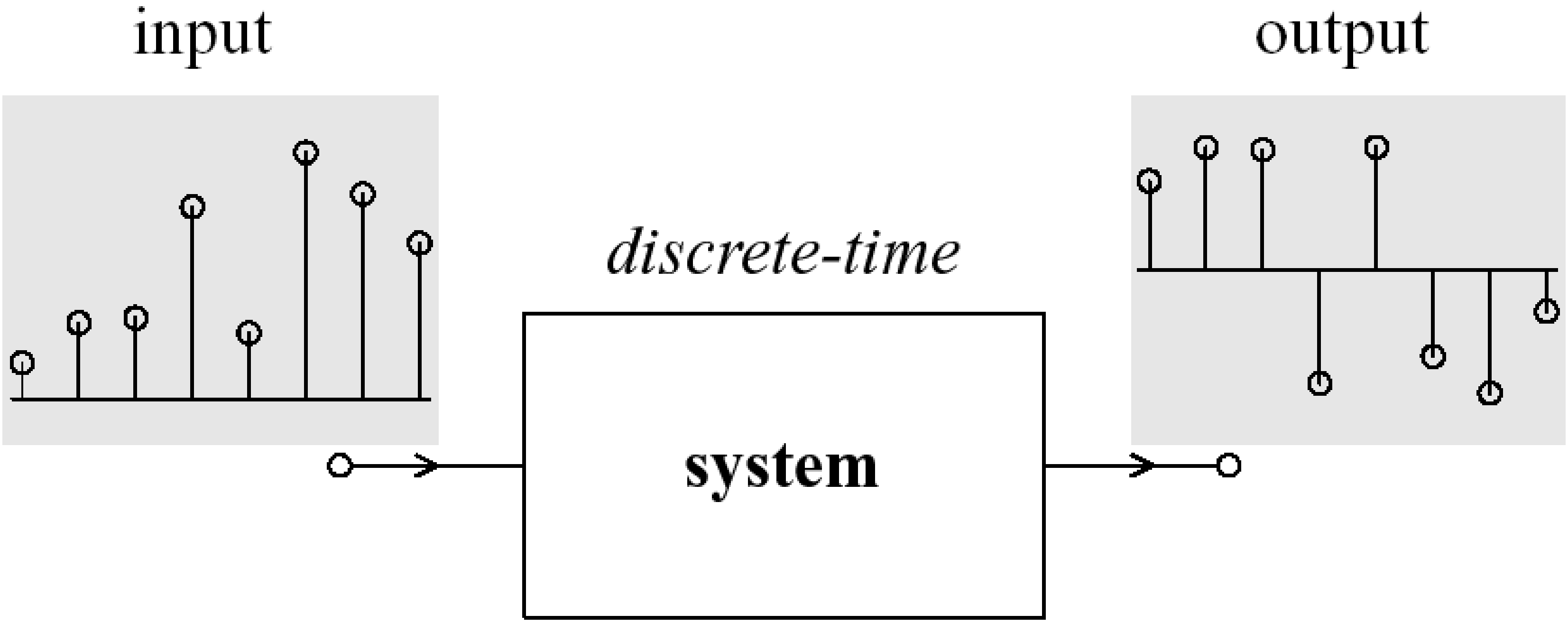
- System **analysis**: analyze the output signal when input signal and system is given.
- System **synthesis**: design the system when input and output signal is given.

Continuous-time system



Continuous-time system: the input and output signals are continuous time

Discrete-time system



Discrete-time system has discrete-time input and output signals

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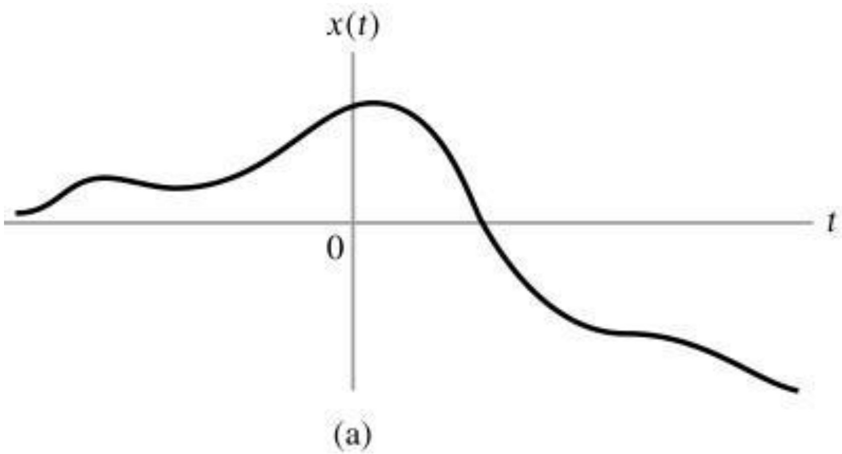
Signal classification

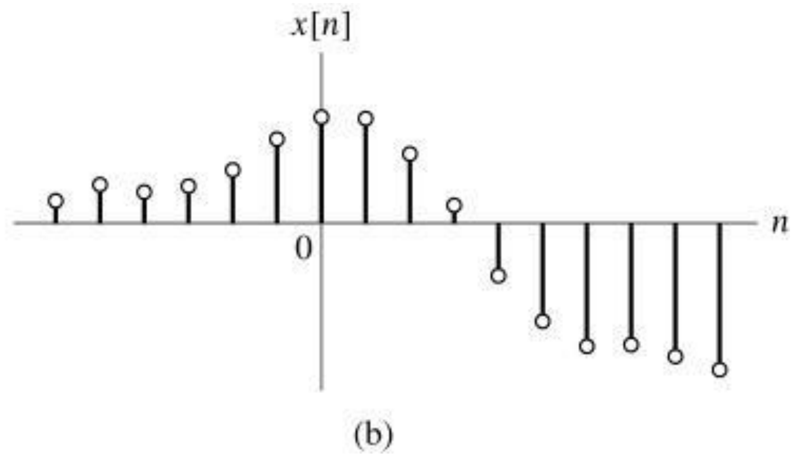
Signal classification	
Continous time	Discrete time
Even	Odd
Periodic	Nonperiodic/aperiodic
Deterministic	Random
Energy	Power

Continuous & discrete time signal

- *x(t)* is defined for all time *t*.
- *x[n]* is defined only at discrete instants of time.
- $\boldsymbol{x[n] = x(nT_s)}$, $n = 0, \pm1, \pm2, \pm3, \ldots$
- T_s: sampling period



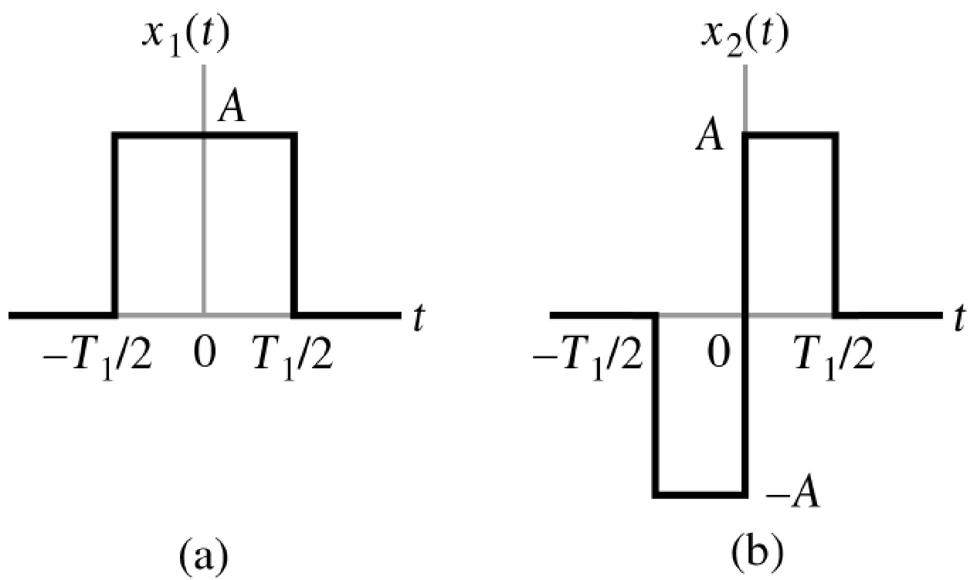
(a) Continuous-time signal $x(t)$.



(b) Representation of $x(t)$ as a discrete-time signal $x[n]$.

Even & odd signal

- Even signal (symmetric about vertical axis)
  - x(-t) = x(t) for all t.
- Odd signal (asymmetric about vertical axis)
  - x(-t) = -x(t) for all t.



Even & odd signal (example)

- Consider the signal

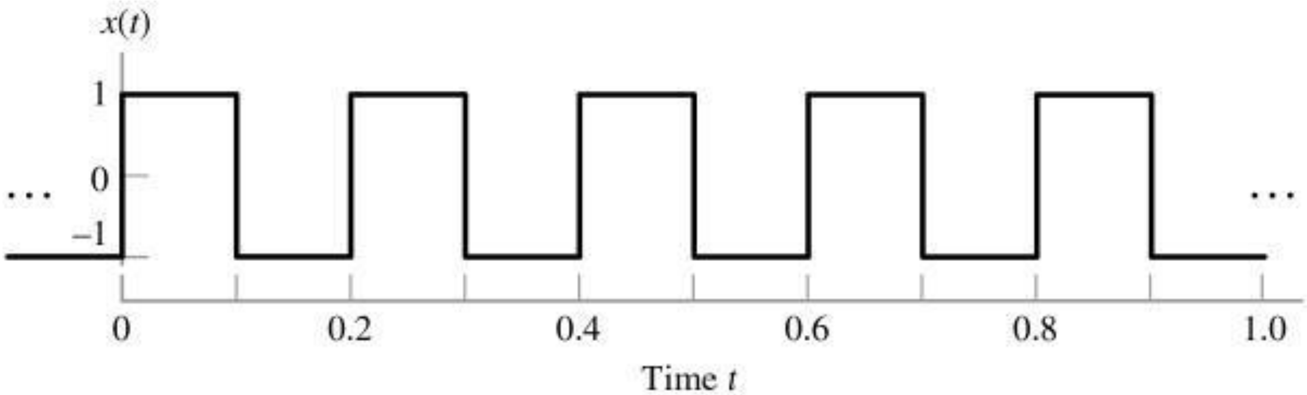
$$x(t) = \begin{cases} \sin(\frac{\pi t}{T}), -T \le t \le T \\ 0, \quad \text{otherwise} \end{cases}$$

- Is the signal x(t) an even or an odd function of time t?
- Clue: replace *t* with *–t*
- Answer: odd signal because x(-t) = -x(t)

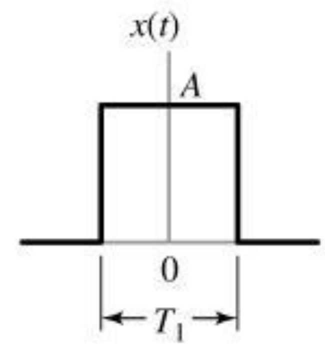
Periodic & nonperiodic signals

- Periodic signal
  - $x(t) = x(t+T)$, for all t
  - T = fundatamental period
  - Fundamental frequency, $f = 1/T$ unit Hz
  - Angular frequency, $\omega = 2\pi f$ unit rad/s
- Nonperiodic signal
  - No value of T satisties the condition above

- (a) Periodic signal
- (b) Nonperiodic signal



(a)

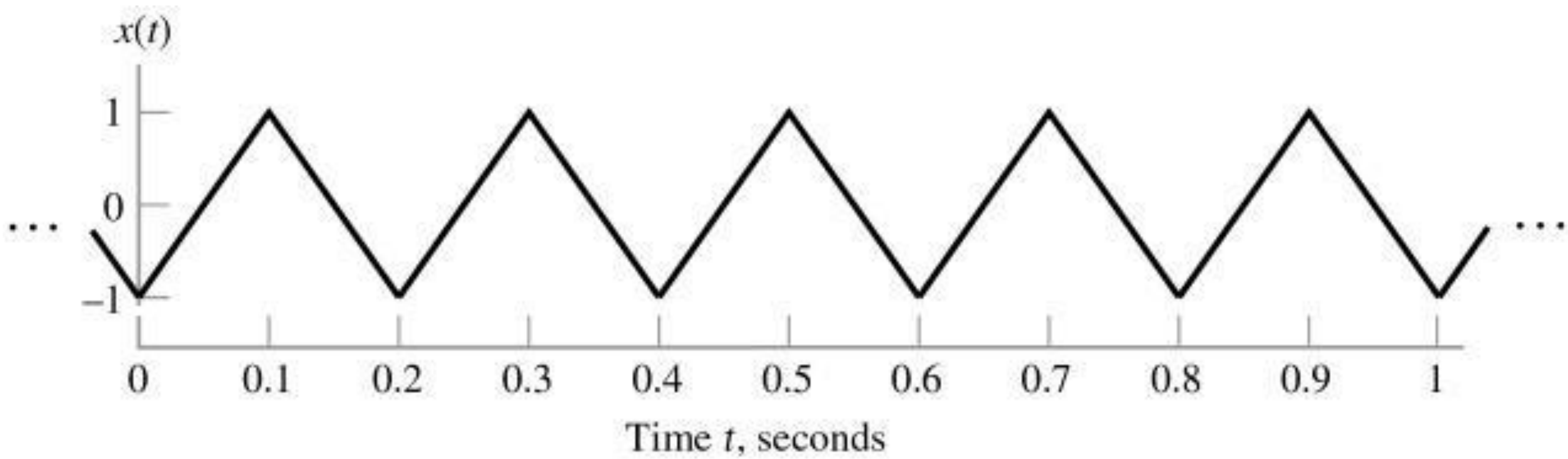


(b)

- For (a), find the amplitude and period of x(t)

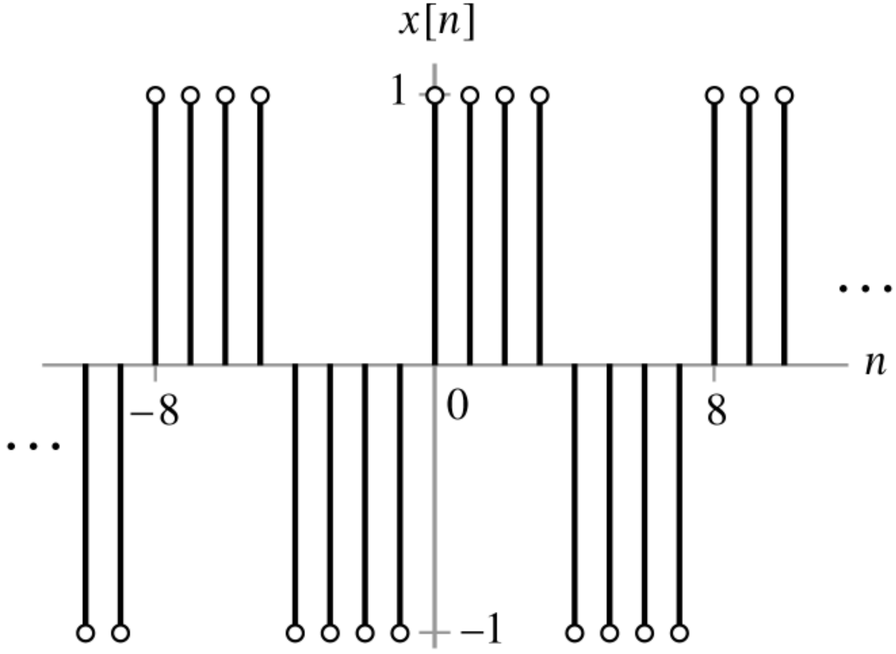
(example)

- What is the fundamental frequency of triangular wave below? Express the fundamental frequency in units of Hz and rad/s.

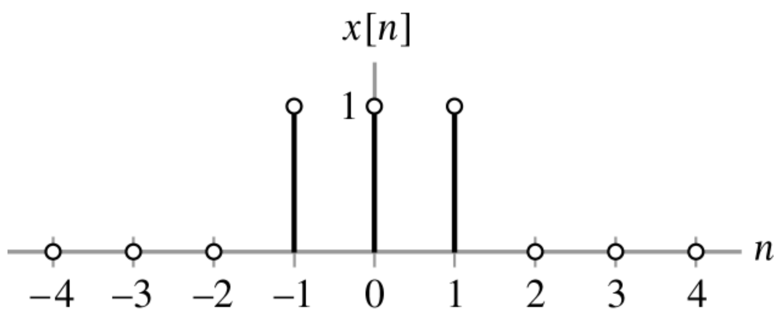


Periodic & nonperiodic signal for discrete time signal

- Periodic discrete time signal
  - x[n] = x[n + N],   for integer n



Periodic signal



Nonperiodic signal

- For each of the following signals, determine whether it is periodic, and if it is, find the fundamental period.
  - $x(t) = \cos^2(2\pi t)$
  - $x(t) = \sin^3(2t)$
  - $x[n] = (-1)^n$
  - $x[n] = \cos(2n)$
  - $x[n] = \cos(2\pi n)$

T = 0.5 s, T = π s, T = 2 sample, nonperiodic, T = 1 sample

Deterministic & random signal

- Deterministic signal: there is no uncertainty with respect to its value at any time. *Specified function*.
- Random signal: there is uncertainty before it occurs.

Energy & power signals

- Energy signal; $0 < E < \infty$
- Power signal; $0 < P < \infty$

Continuous time signals

$$E = \int_{-\infty}^{\infty} x^2(t)dt$$

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt$$

Discrete time signals

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

$$P = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

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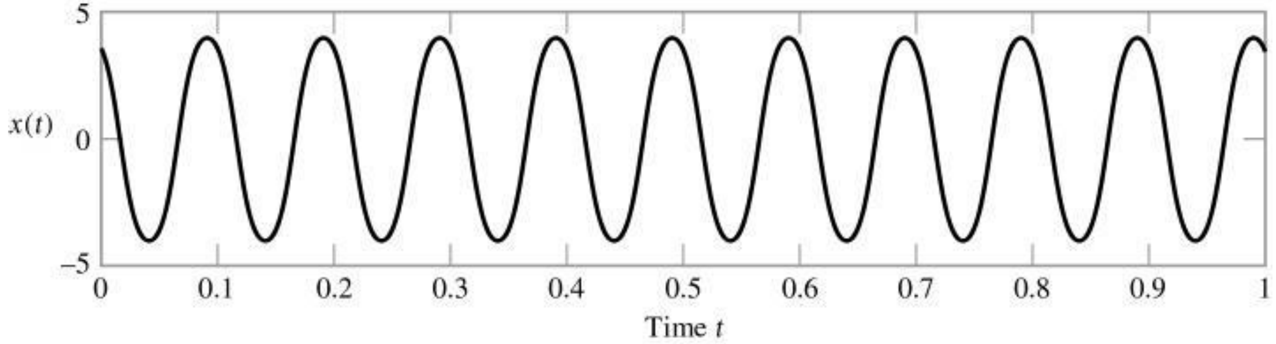
Useful signal models

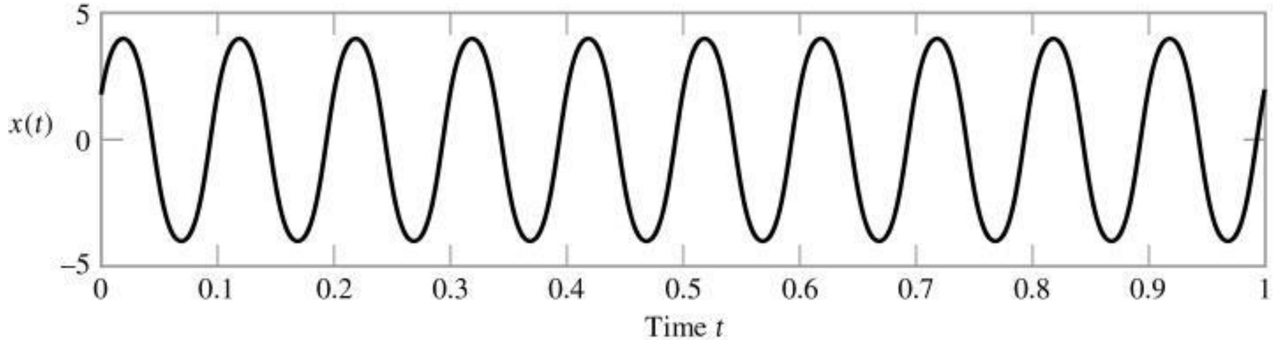
- Sinusoidal
- Exponential
- Unit step function
- Unit impulse function

Sinusoidal

- (a) Sinusoidal signal $A\cos(\omega t + \Phi)$ with phase $\Phi = +\pi/6$ radians.
  (b) Sinusoidal signal $A\sin(\omega t + \Phi)$ with phase $\Phi = +\pi/6$ radians.



(a)



(b)

Exponential

$$x_e(t) = X_e\,\mathrm{e}^{bt}$$

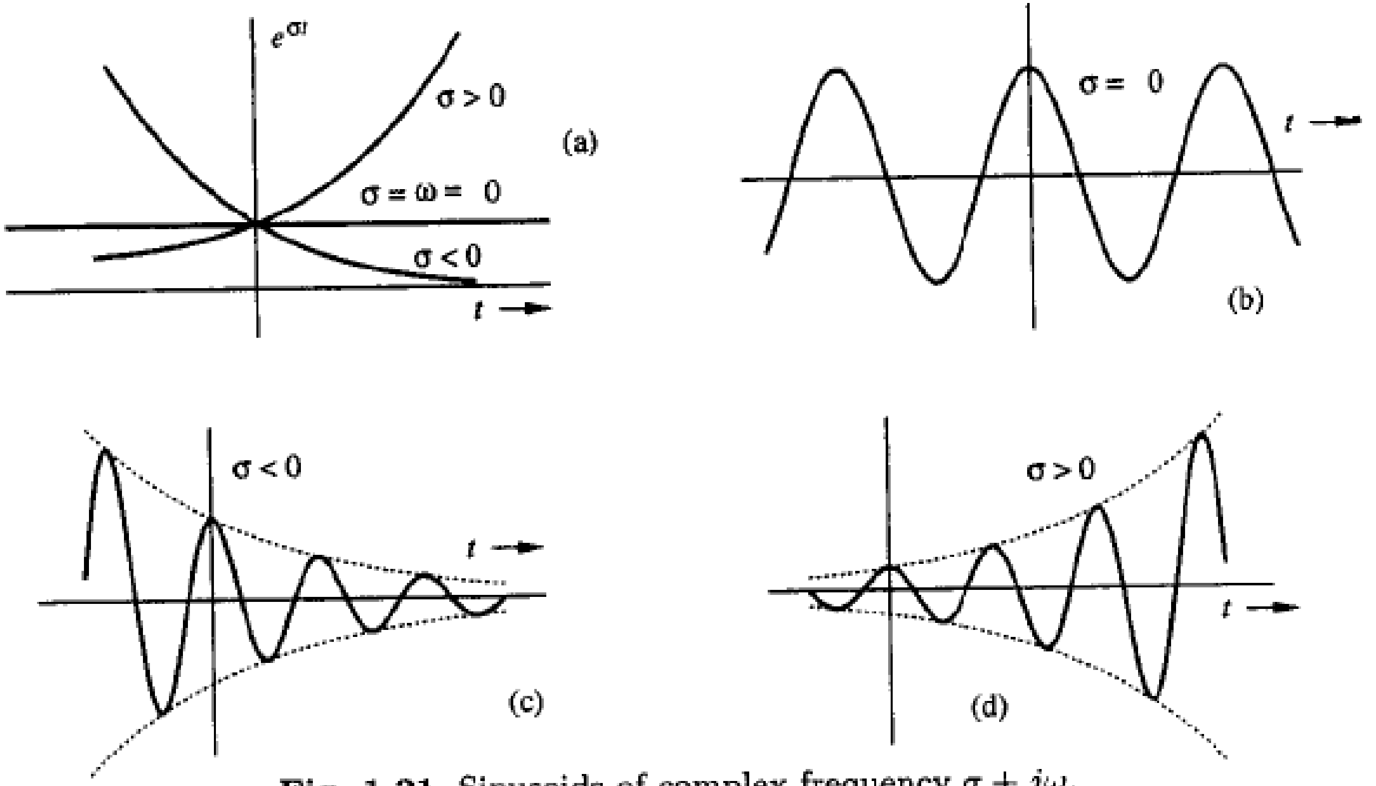
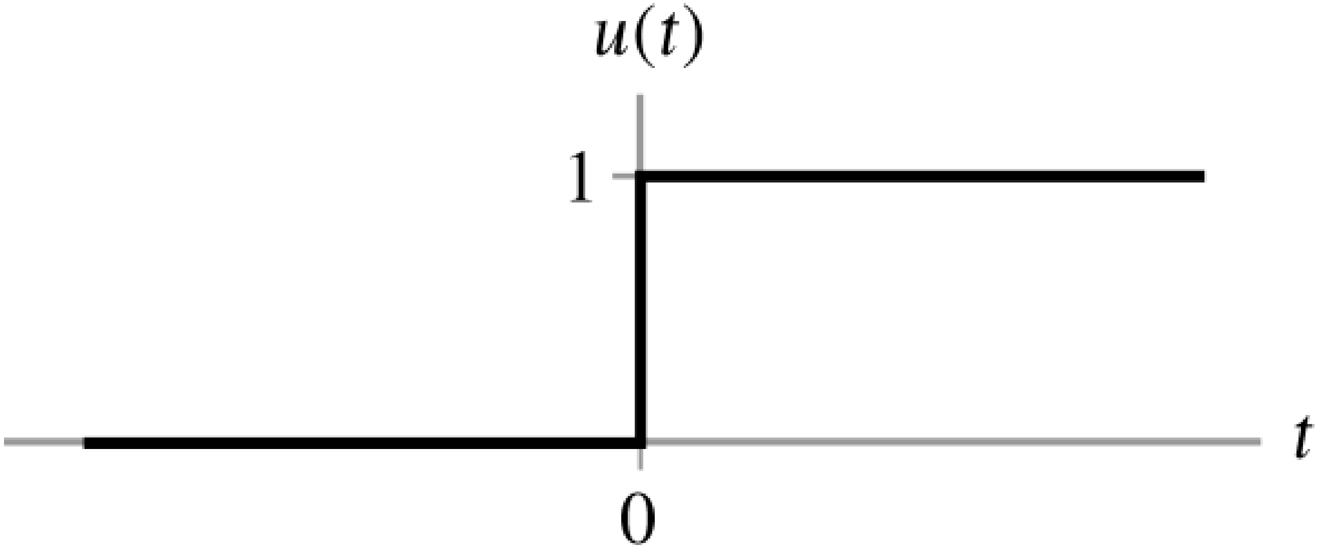


Fig. 1.21 Sinusoids of complex frequency $\sigma + j\omega$.

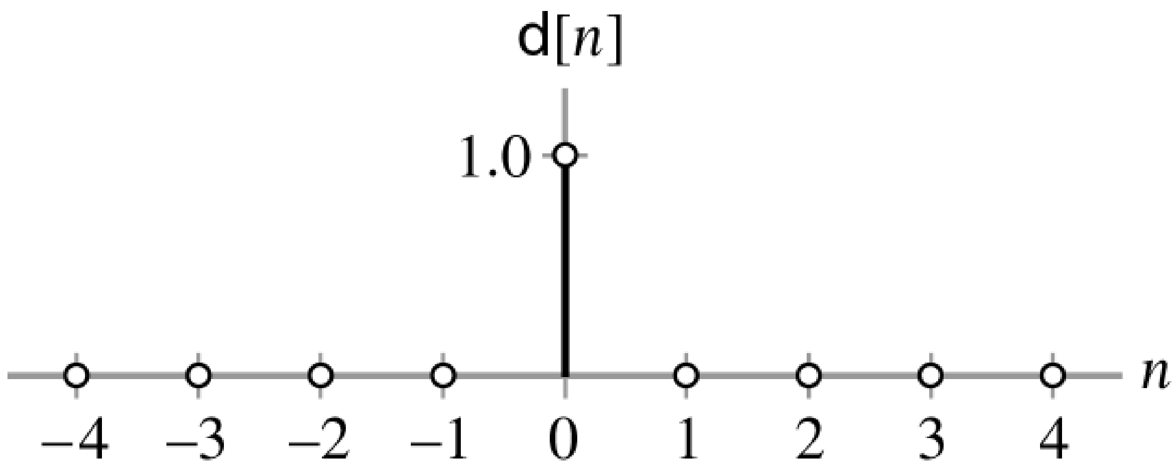
Unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$



Unit impulse function

- Pulse signal = $$p_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 < t \le \varepsilon \\ 0, & \text{otherwise} \end{cases}$$
- Unit impulse (Dirac delta) = $$\delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t)$$ $$\delta(t) = 0, \quad t \ne 0$$ $$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$



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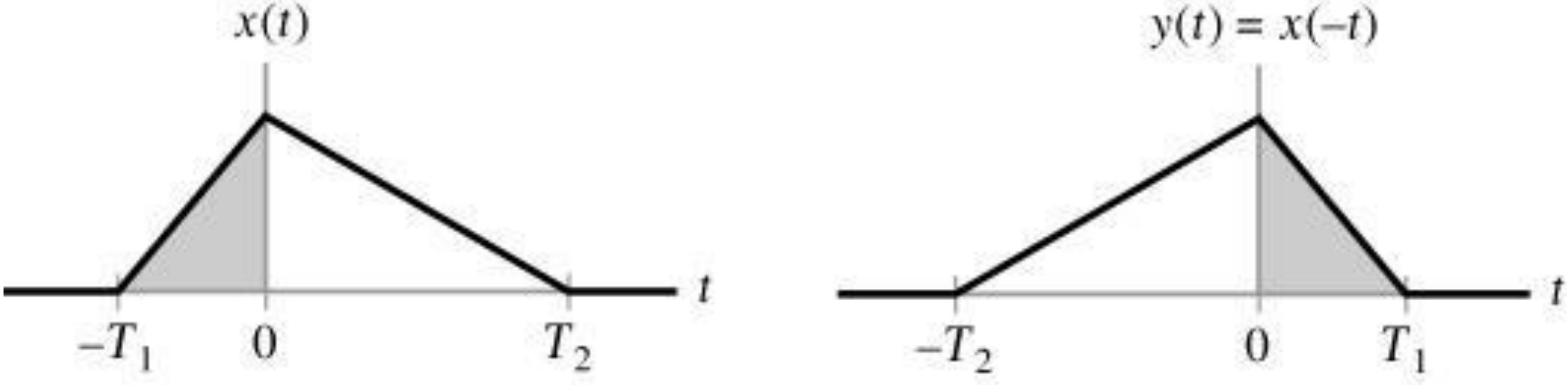
Operation on signal

- Dependent variable: x, y, etc…
  - Multiplication
  - Addition
  - Substraction
  - Integration
  - Differentiation
- Independent variable: (t) etc…
  - Time flip / reflection / time reverse
  - Time scale
  - Time shift

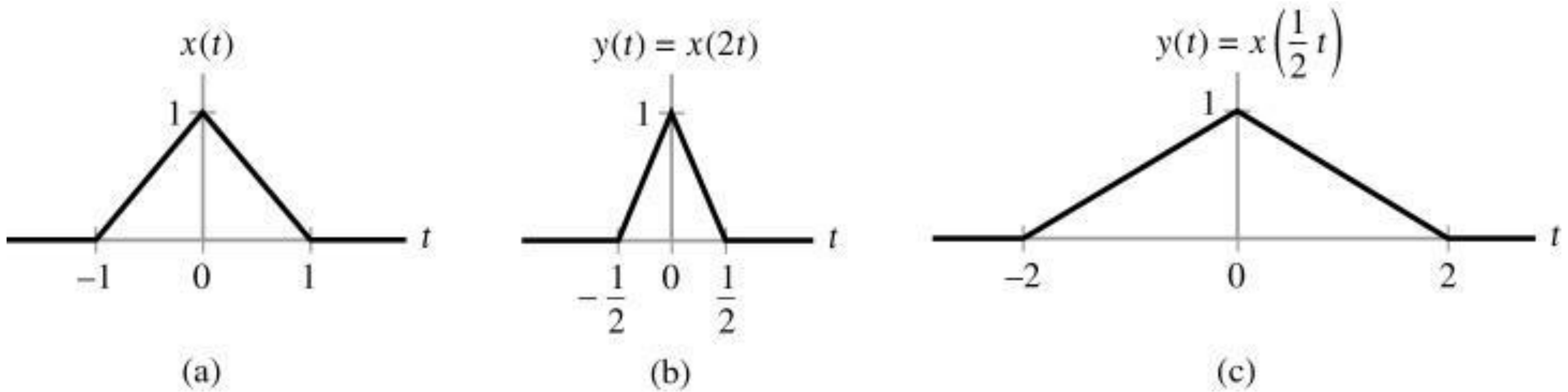
Time flip/reflection

- Operation of reflection: (a) continuous-time signal $x(t)$ and (b) reflected version of $x(t)$ about the origin.

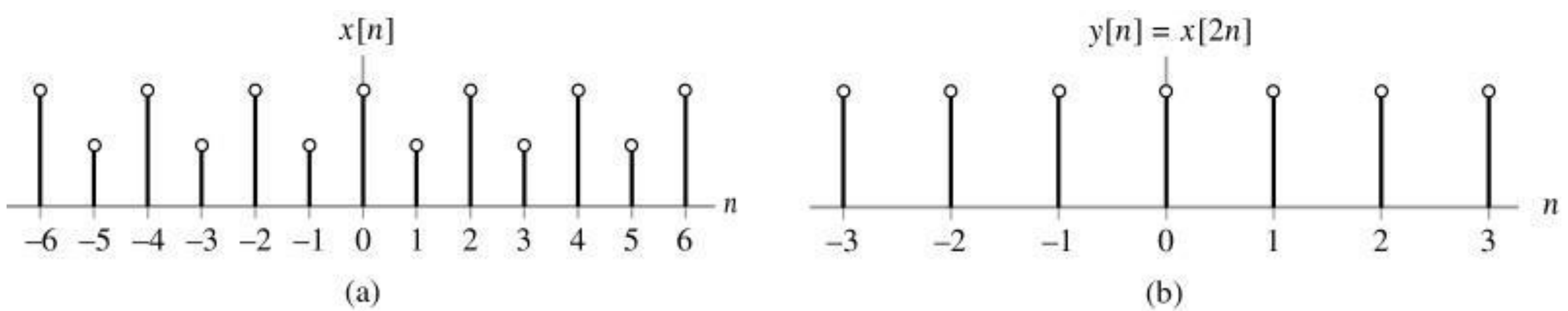


(a) (b)

Time scale



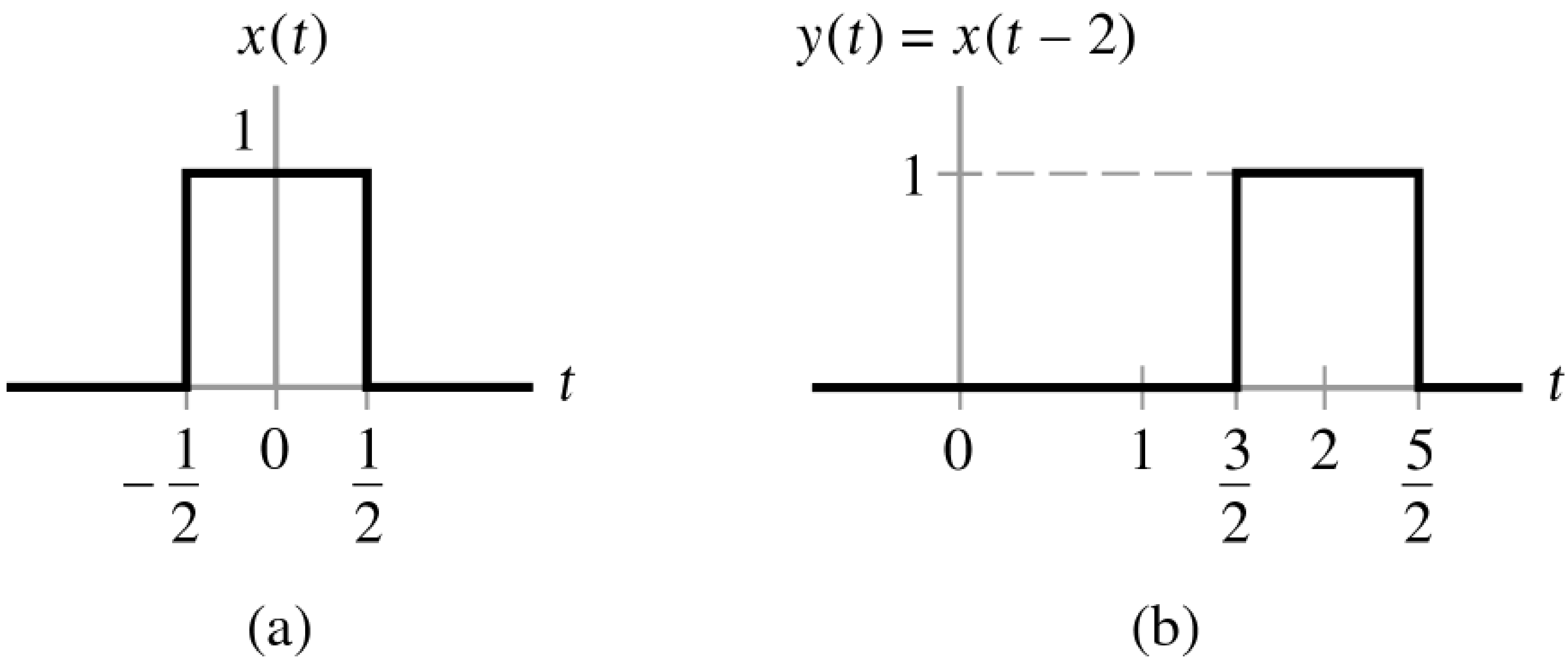
Time scale on continuous signal



Time scale on discrete signal

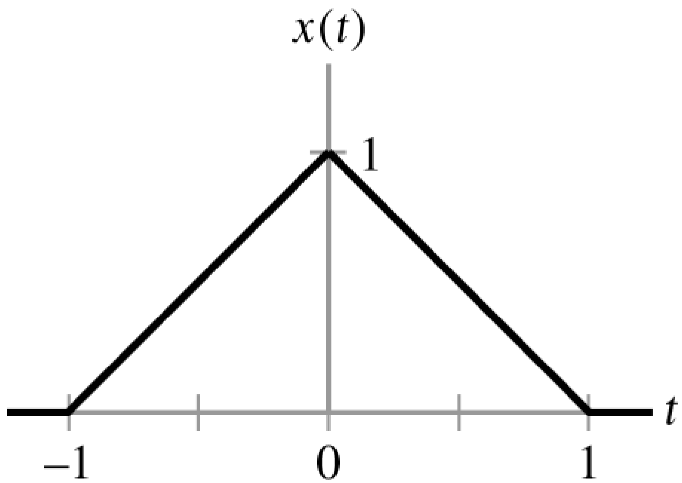
Time shift

- Time-shifting operation: (a) continuous-time signal in the form of a rectangular pulse of amplitude 1.0 and duration 1.0, symmetric about the origin; and (b) time-shifted version of *x*(*t*) by 2 time shifts.



Exercise of signal operation

- Suppose x(t) is a triangular signal



- Find

(a) x(3t)

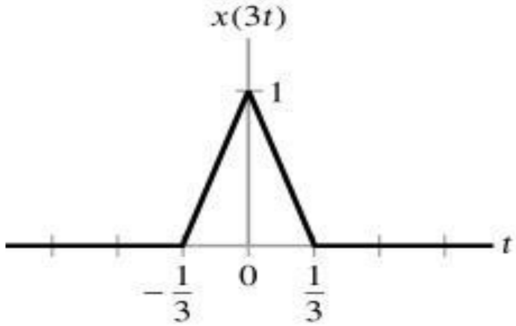
(b) x(3t+2)

(c) x(-2t-1)

(d) x(2(t+2))

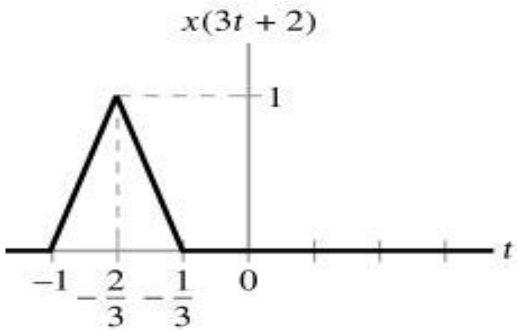
(e) x(2(t-2))

(f) x(3t) + x(3t+2)



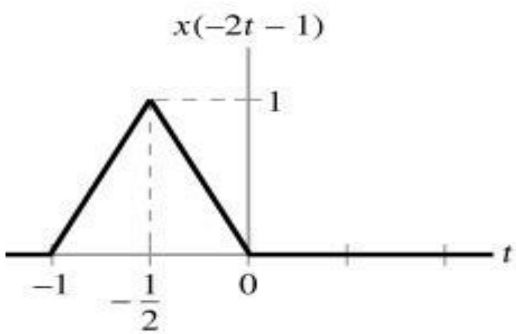
x(3t)
1
t
−1/3
0
1/3

(a)



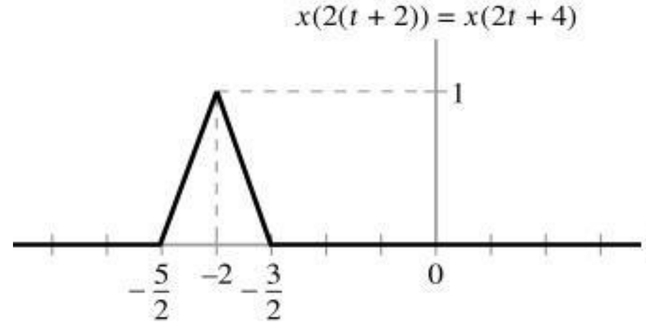
x(3t + 2)
1
t
−1
−2/3
−1/3
0

(b)



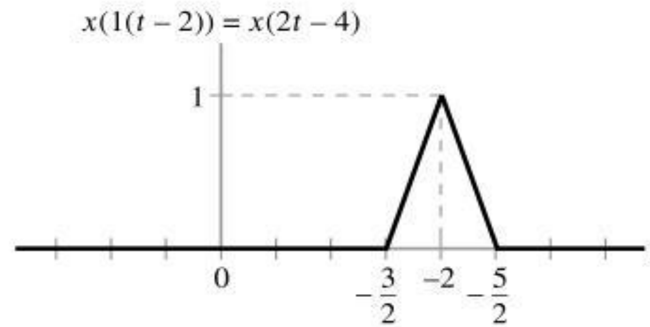
x(−2t − 1)
1
t
−1
−1/2
0

(c)



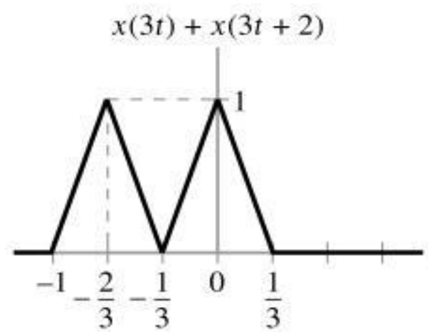
x(2(t + 2)) = x(2t + 4)
1
t
−5/2
−2
−3/2
0

(d)



x(1(t − 2)) = x(2t − 4)
1
t
0
3/2
2
5/2

(e)



x(3t) + x(3t + 2)
1
t
−1
−2/3
−1/3
0
1/3

(f)

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Properties of system

- Memory
- Stability
- Invertibility
- Causality
- Linearity
- Time-invariance

Memory vs. Memoryless Systems

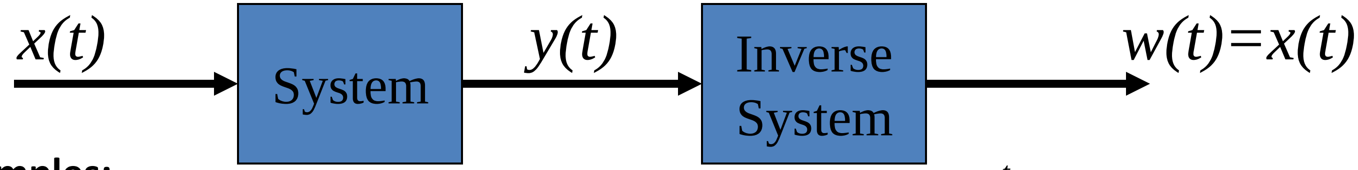
- *Memoryless* (or *static*) Systems: System output $y(t)$ depends only on the input at time t, i.e. y(t) is a function of $x(t)$.
- *Memory* (or *dynamic*) Systems: System output $y(t)$ depends on input at past or future of the current time t, i.e. $y(t)$ is a function of $x(\tau)$ where $-\infty < \tau < \infty$.
- Examples:
  - A resistor: $y(t) = R\,x(t)$
  - A capacitor: $$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$
  - A one unit delayer: $y[n] = x[n-1]$
  - An accumulator: $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Stability and Invertibility

- **<u>Stability</u>: A system is stable if it results in a bounded output for any bounded input, i.e. bounded-input/bounded-output (BIBO).**
  - **If $|x(t)| < k_1$, then $|y(t)| < k_2$.**
  - **Example:**

$$y(t) = \int_0^t x(t)dt \qquad y[n] = 100x[n]$$

- **<u>Invertibility</u>: A system is invertible if distinct inputs result in distinct outputs. If a system is invertible, then there exists an "inverse" system which converts output of the original system to the original input.**



  - **Examples:**

$$y(t) = 4x(t) \qquad y[n] = \sum_{k=-\infty}^{n} x[k] \qquad y(t) = \int_{-\infty}^{t} x(t)dt$$

$$w(t) = \frac{1}{4}y(t) \qquad w[n] = y[n] - y[n-1] \qquad w(t) = \frac{dy(t)}{dt}$$

Causality

- A system is called ***causal*** if the output depends only on the present and past values of the input

Linearity

- A system is linear if it satisfies the properties:
  - It is *additivity*: $x(t) = x_1(t) + x_2(t) \Rightarrow y(t) = y_1(t) + y_2(t)$
  - And it is *homogeneity* (or *scaling*): $x(t) = a\, x_1(t) \Rightarrow y(t) = a\, y_1(t)$, for a any complex constant.

- The two properties can be combined into a single property:
  - Superposition:

$$x(t) = a\, x_1(t) + b\, x_2(t) \Rightarrow y(t) = a\, y_1(t) + b\, y_2(t)$$

$$x[n] = a\, x_1[n] + b\, x_2[n] \Rightarrow y[n] = a\, y_1[n] + b\, y_2[n]$$

Time-Invariance

- A system is time-invariant if a delay (or a time-shift) in the input signal causes the same amount of delay (or time-shift) in the output signal, i.e.:

  $x(t) = x_1(t-t_0) \Rightarrow y(t) = y_1(t-t_0)$

  $x[n] = x_1[n-n_0] \Rightarrow y[n] = y_1[n-n_0]$

Time and frequency domains

- Most analysis were done in frequency domain.
- Much more information can be extracted from a signal in frequency domain.
- To represent a signal in frequency domain, some method were introduced, the first one is
- FOURIER SERIES…

Z-Transform

Introduction

- *The Laplace Transform (s domain) is a valuable tool for representing, analyzing & designing continuos-time signals & systems.*
- *The z-transform is convenient yet invaluable tool for representing, analyzing & designing discrete-time signals & systems.*
- *The resulting transformation from s-domain to z-domain is called z-transform.*
- *The relation between s-plane and z-plane is described below :*

$$z = e^{sT}$$

- *The z-transform maps any point $s = \sigma + j\omega$ in the s-plane to z-plane ($r\angle\theta$).*

Z-Transform

For continuous-time signal,

$$x(t) \rightleftharpoons X(s)$$

Time Domain → $x(t)$; S-Domain → $X(s)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

For discrete-time signal,

$$\int \cdot \Rightarrow \sum \cdot \quad , t \Rightarrow nT \quad \Rightarrow \sum x(nT)\, e^{-sT n}$$

$$e^{sT} = z$$

$$x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftharpoons}} X(z)$$

Time Domain → $x(n)$; Z-Domain → $X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \; ; = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Causal System → $\sum_{n=0}^{\infty} x(n) z^{-n}$

where,

$$z = e^{sT} \quad , s = \sigma + j\omega \; , T = \text{sampling time} = \frac{1}{f_s}$$

Z-Transform Definition

- *The z-transform of sequence x(n) is defined by*

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Two sided z transform
Bilateral z transform

> ➢ *For causal system*

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

One sided z transform
Unilateral z transform

- *The z transform reduces to the Discrete Time Fourier transform (DTFT) if r=1; $z = e^{-j\omega}$.*

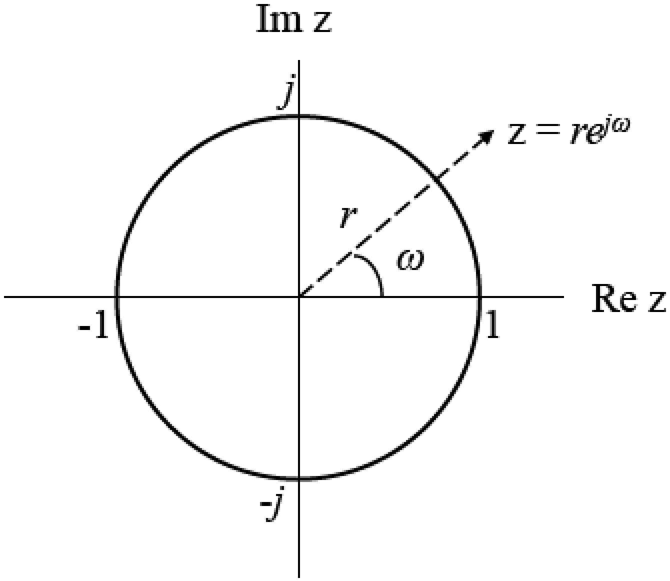
DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Geometrical interpretation of z-transform

- *The point* $z = re^{j\omega}$ *is a vector of length r from origin and an angle* ω *with respect to real axis.*

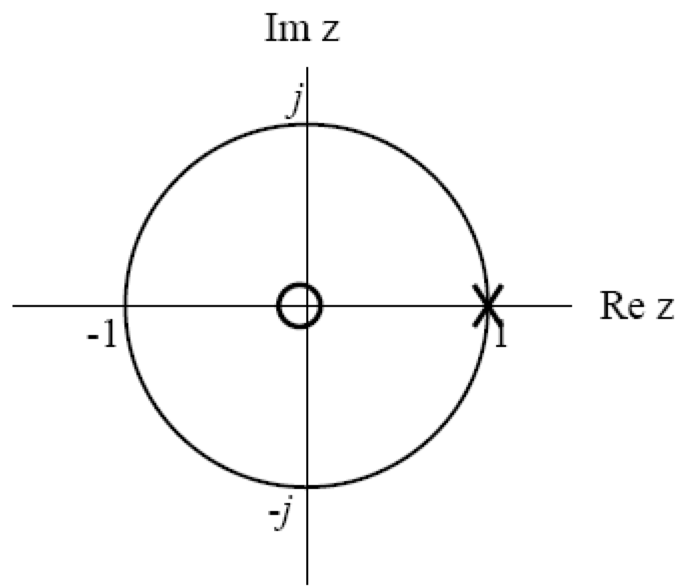
- *Unit circle : The contour* $|z| = 1$ *is a circle on the z-plane with unity radius*



DTFT is to *evaluate z-transform on a unit circle.*

Pole-zero Plot

- *A graphical representation of z-transform on z-plane*
  - *Poles denote by "x" and*
  - *zeros denote by "o"*



Example

- *Find the z-transform of,* $a)\ \delta(n)$ $b)\ u(n)$

✓ *Solution:*

$$a)\ Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = \delta(0) z^{-0} = \delta(0) = 1$$

$$b)\ Z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

It's a geometric sequence $a = 1,\quad r = z^{-1},\quad n = \infty$

$$Z\{u(n)\} = \frac{1 - z^{-\infty}}{1 - z^{-1}}, \qquad |z| > 1$$

$$= \frac{z}{z - 1}$$

<u>Recall</u>: Sum of a Geometric Sequence

$$S = a\,\frac{1 - r^n}{1 - r}$$

where, a: first term, r: common ratio, n: number of terms

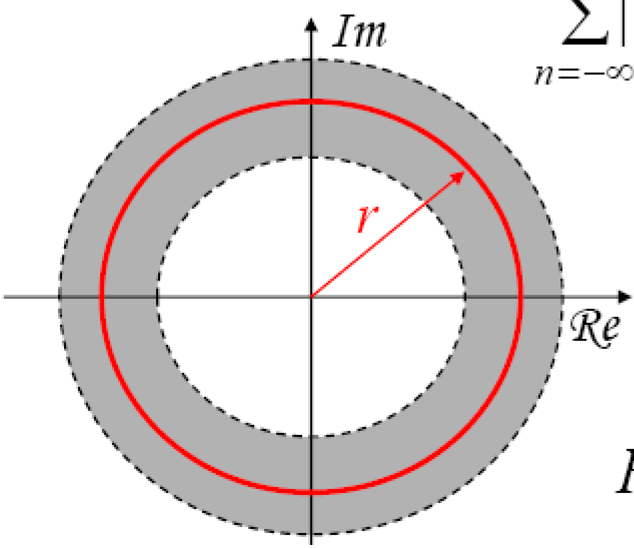
Region Of Convergence (ROC)

- *ROC of X(z) is the set of all values of z for which X(z) attains a finite value.*

- *Give a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.*

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$



ROC is an annual ring centered on the origin.

$$R_{x-} < |z| < R_{x+}$$

$$ROC = \{z = re^{j\omega} \mid R_{x-} < r < R_{x+}\}$$

Ex. 1 *Find the z-transform of the following sequence*

$$x = \{2, -3, 7, 4, 0, 0, \ldots\ldots\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 2 - 3z^{-1} + 7z^{-2} + 4z^{-3}$$

$$= \frac{2z^3 - 3z^2 + 7z + 4}{z^3}, \quad |z| > 0$$

The ROC is the entire complex z - plane except the origin.

Ex. 2 *Find the z-transform of $\delta[n]$*

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

with an ROC consisting of the entire z - plane.

Ex. 3 *Find the z-transform of δ [n -1]*

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1} = \frac{1}{z}$$

with an ROC consisting of the entire z - plane except $z = 0$.

Ex. 4 *Find the z-transform of δ [n +1]*

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z$$

with an ROC consisting of the entire z - plane except $z = \infty$,

i.e., there is a pole at infinity.

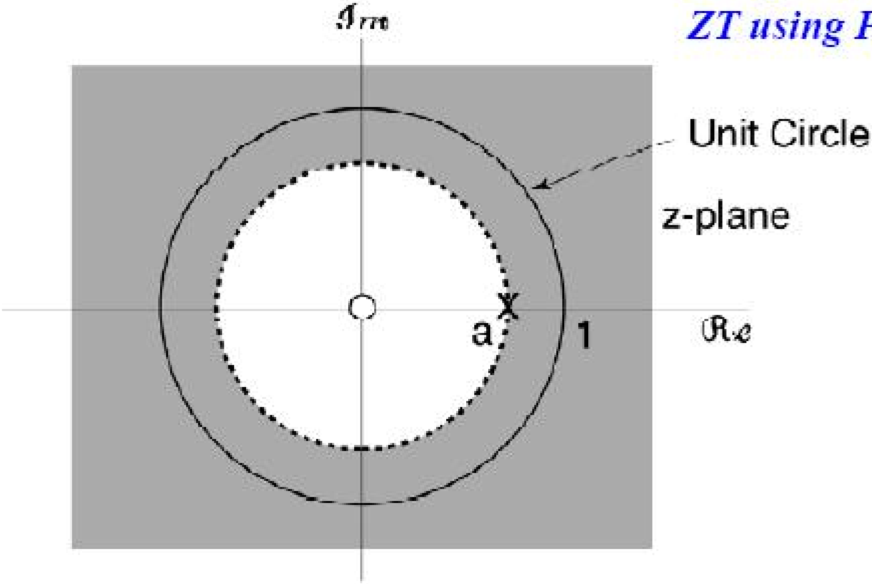
Ex.5 *Find the z-transform of the following right-sided sequence (causal)*

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

This form to find inverse ZT using PFE

This form to find pole and zero locations



If $|az^{-1}| < 1$, i.e., $|z| > |a|$

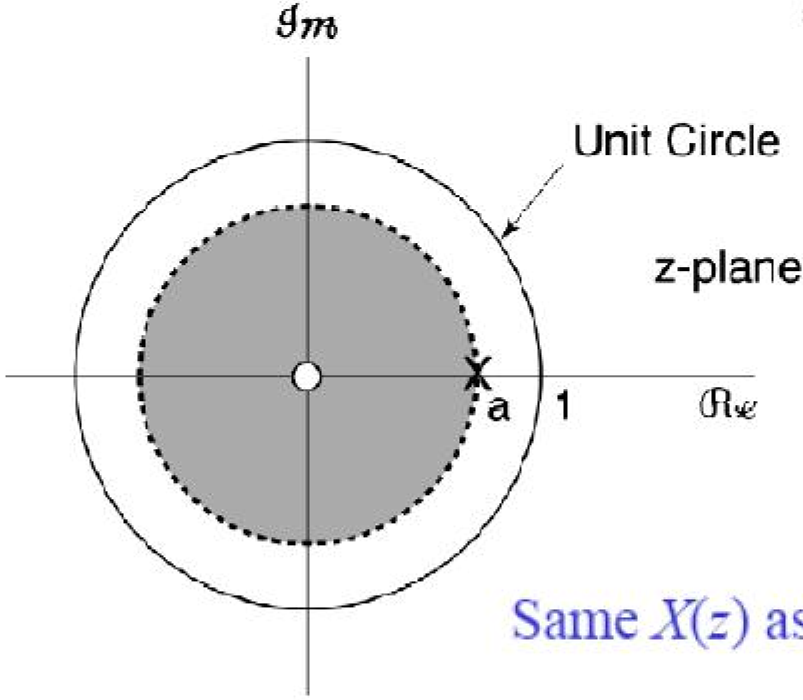
That is, ROC $|z| > |a|$, outside a circle

Ex.6 *Find the z-transform of the following left-sided sequence*

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$



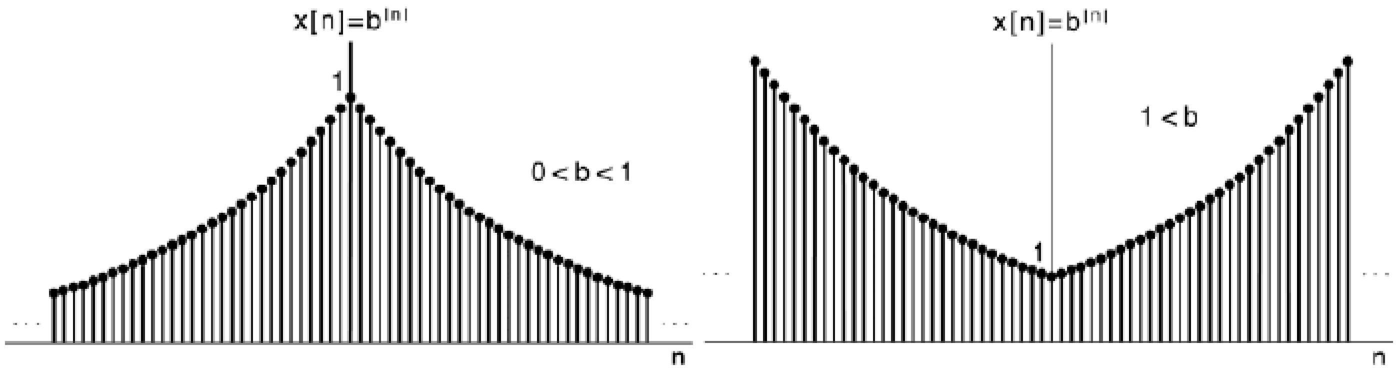
$$= 1 - \frac{1}{1 - a^{-1}z} = \frac{a^{-1}z}{a^{-1}z - 1}$$

$$= \frac{z}{z-a},$$

If $|a^{-1}z| < 1$, $i.e., |z| < |a|$

Same $X(z)$ as in **Ex #1**, but different ROC.

Ex. 7 *Find the z-transform of* $x[n] = b^{|n|}, \quad b > 0$



Rewriting x[n] as a sum of left-sided and right-sided sequences and finding the corresponding z-transforms,

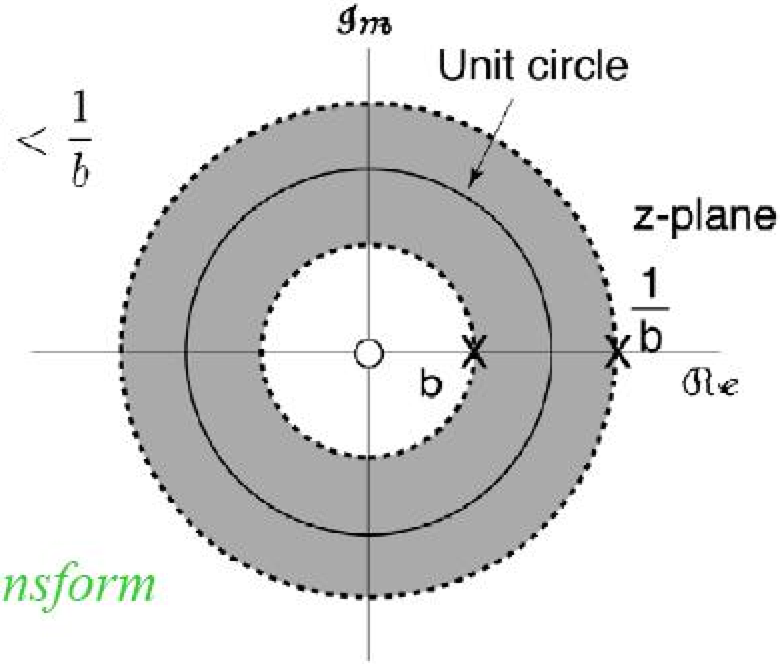
$$x[n] - b^n u[n] + b^{-n} u[-n-1]$$

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1}z^{-1}} \quad , \quad b < |z| < \frac{1}{b}$$

where

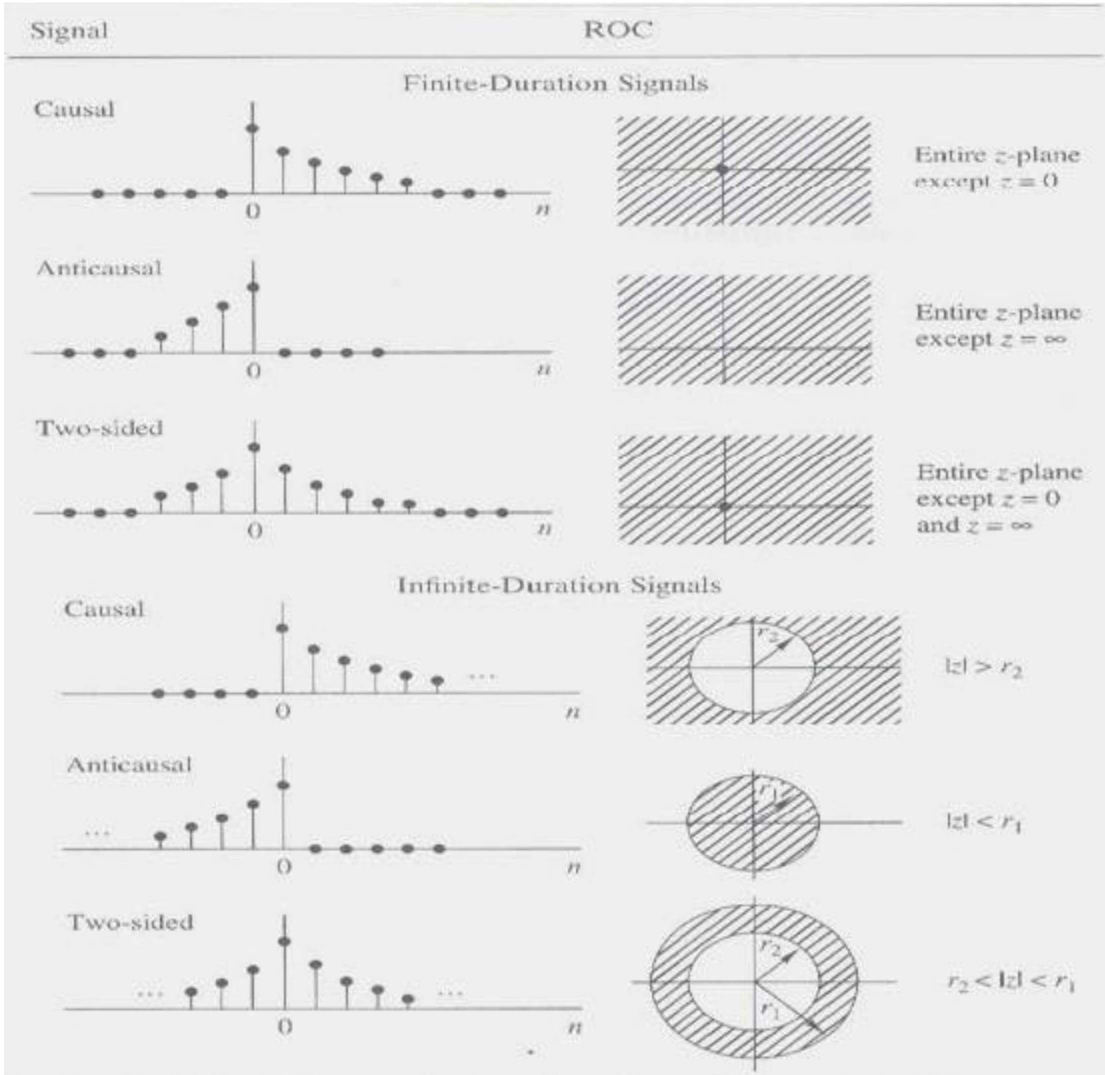
$$b^{-n}u[-n-1] \longleftrightarrow \frac{-1}{1 - b^{-1}z^{-1}}, \quad |z| < \frac{1}{b}$$

$$b^{n}u[n] \longleftrightarrow \frac{1}{1 - bz^{-1}}, \quad |z| > b$$



Notice from the ROC that the z-transform doesn't exist for b > 1

Characteristic Families of Signals with Their Corresponding ROC



Properties of ROC

- *A ring or disk in the z-plane centered at the origin.*
- *The Fourier Transform of x(n) is converge absolutely iff the ROC includes the unit circle.*
- *The ROC cannot include any poles*
- *Finite Duration Sequences: The ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.*
- *Right sided sequences (causal seq.): The ROC extends outward from the outermost finite pole in $X(z)$ to $z=\infty$.*
- *Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to $z=0$.*
- *Two-sided sequence: The ROC is a ring bounded by two circles passing through two pole with no poles inside the ring*

Properties of z-Transform

(1) *Linearity :* $ax[n]+by[n] \longleftrightarrow aX(z)+bY(z)$

(2) Time Shifting $x[n-n_0] \longleftrightarrow z^{-n_0}X(z),$

(3) z-Domain Differentiation $nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz},$

(4) Z - scale Property : $a^n x[n] \longleftrightarrow X\left(\dfrac{z}{a}\right)$

(5) *Time Reversal :* $x[-n] \longleftrightarrow X(\dfrac{1}{z})$

(6) *Convolution :* $h[n]*x[n] \longleftrightarrow H(z)X(z)$

Transfer Function

Rational z-Transform

For most practical signals, the z-transform can be expressed as a ratio of two polynomials

$$X(z) = \frac{N(z)}{D(z)} = G\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

where

G is scalar gain,

$z_1, z_2, \cdots, z_M$ are the *zeroes* of *X(z)*, i.e., the roots of the numerator polynomial

and $p_1, p_2, \cdots, p_N$ are the *poles* of *X(z)*, i.e., the roots of the denominator polynomial.

Commonly used z-Transform pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$(n+1)\ \alpha^n u[n]$	$\frac{1}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\ u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > \|r\|$
$(r^n \sin \omega_o n)\ [n]$	$\frac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > \|r\|$

Z-Transform & pole-zero distribution & Stability considerations

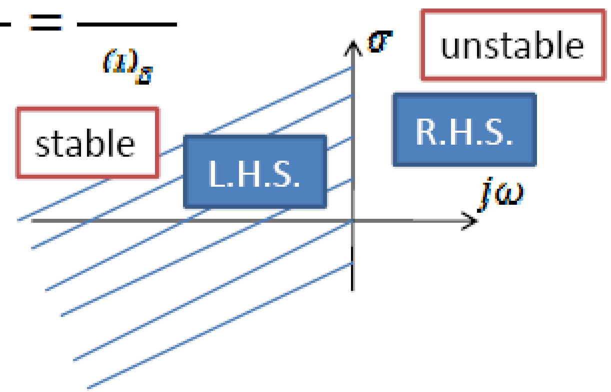
$$\because s = \sigma + j\omega$$

$$\therefore z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$$

Thus, $|z| = e^{\sigma T}$ & $\underline{ang(z)} = \omega T = \frac{2\pi f}{f_s} = \frac{2\pi\omega}{\omega_s}$

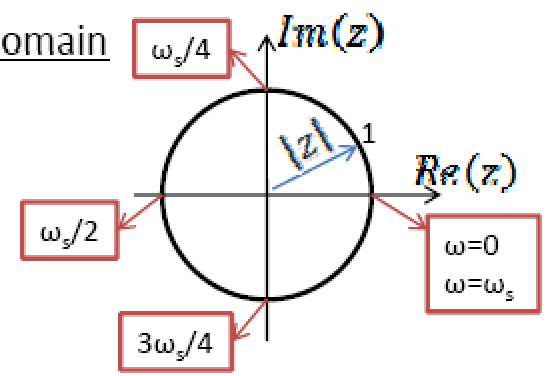
$\angle z$

❖ *Mapping between S-plane & Z-plane is done as follows:*



1) Mapping of Poles on the jω-axis of the s-domain to the z-domain

Maps to a unit circle & represents Marginally stable terms

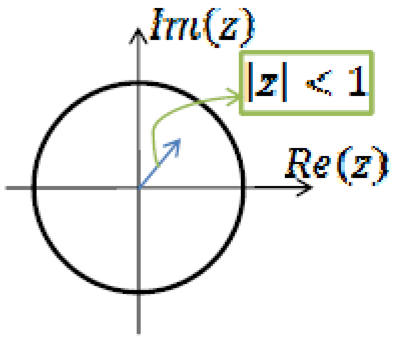


Z-Transform & pole-zero distribution & Stability considerations – cont.

2) Mapping of Poles in the L.H.S. of the s-plane to the z-plane

$$z = e^{\sigma T} e^{j\omega T}, \quad \sigma < 0$$

Maps to inside the unit circle & represents stable terms & the system is stable.



3) Mapping of Poles in the R.H.S. of the s-plane to the z-plane

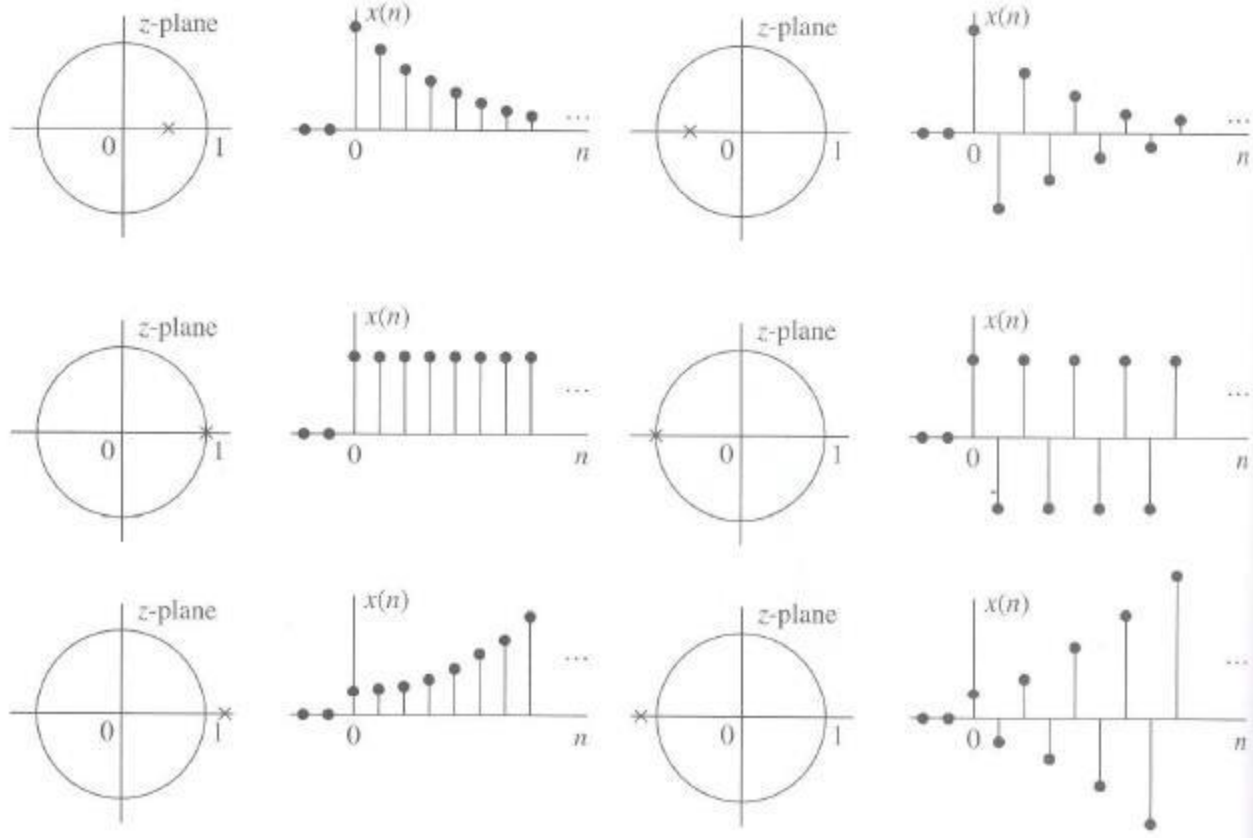
$$z = e^{\sigma T} e^{j\omega T}, \quad \sigma > 0$$

Outside the unit circle & represents unstable terms.

➢ ***Discrete Systems Stability Testing Steps***

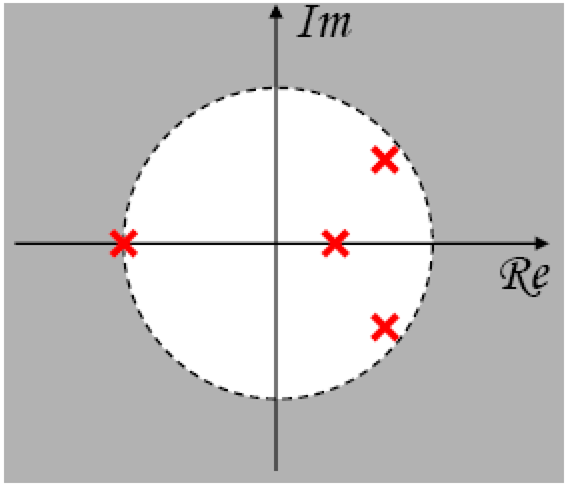
1) *Find the pole positions of the z-transform.*
2) *If any pole is on or outside the unit circle. (Unless coincides with zero on the unit circle) → The system is unstable.*

Pole Location and Time-domain Behavior of Causal Signals

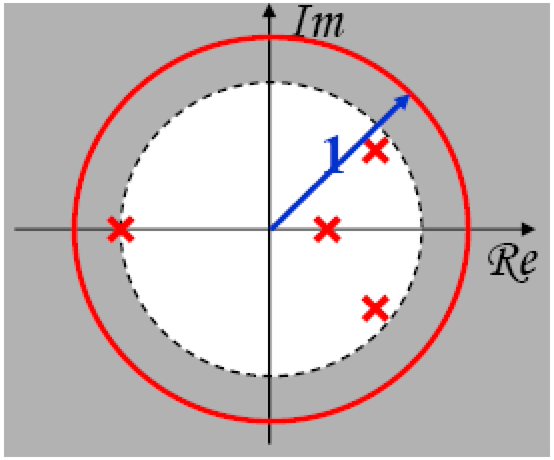


Stable and Causal Systems

Causal Systems : ROC extends outward from the outermost pole.



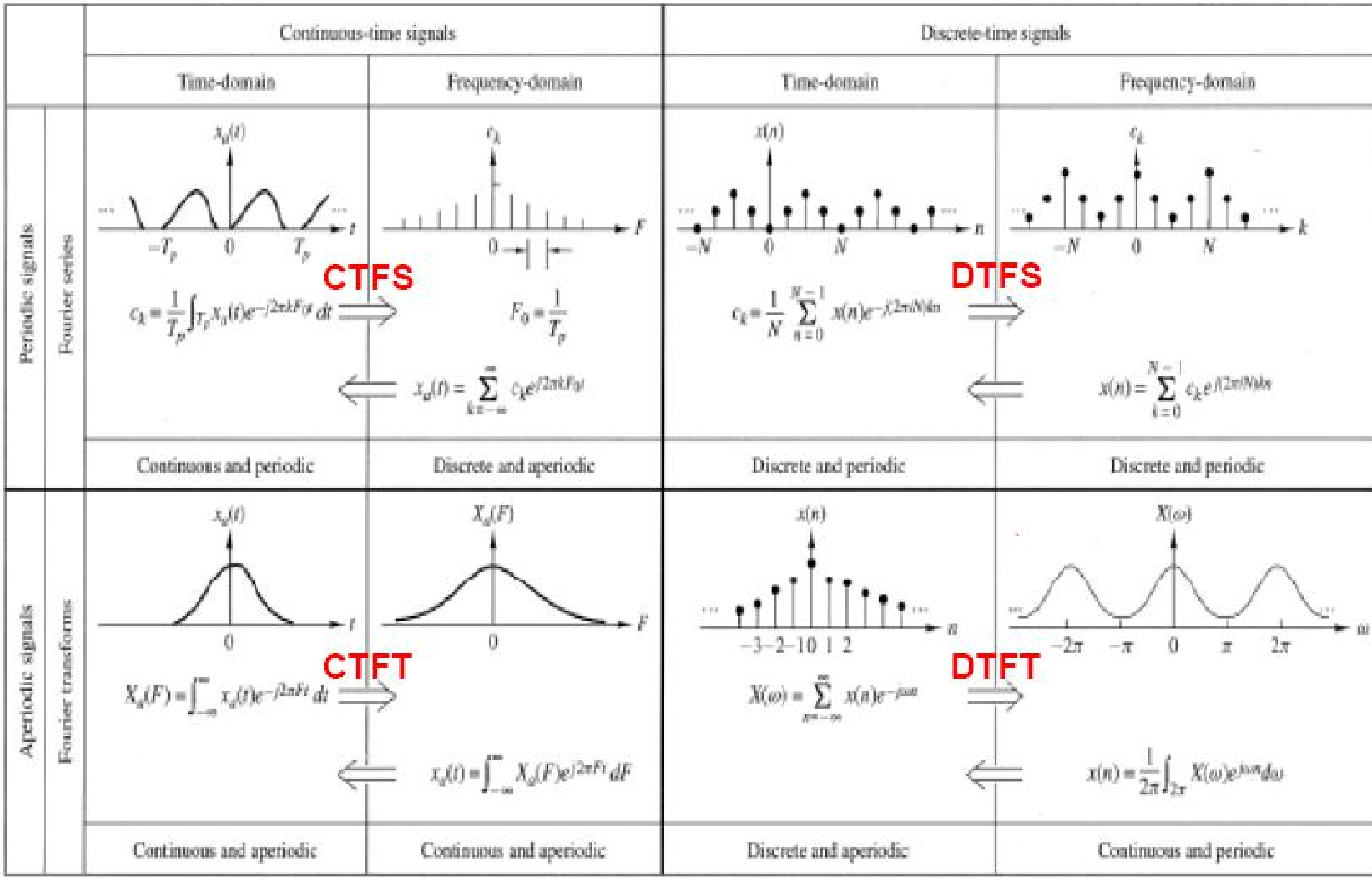
Stable Systems : ROC includes the unit circle.



A stable system requires that its Fourier transform is uniformly convergent.

DFT and FFT

Frequency Domain Vs. Time Domain



Discrete Fourier Transform (DFT)

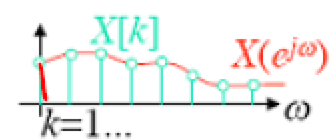
Discrete FT (DFT)	Discrete finite/pdc $x[n]$	Discrete finite/pdc $X[k]$

- *A finite or periodic sequence has only N unique values, x[n] for* $0 \le n \prec N$
- *Spectrum is completely defined by N distinct frequency samples*
- *Divide 0..2π into N equal steps, {ω(k)} = 2πk/N*
- *Uniform sampling of DTFT spectrum:*

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{DTFT}$$

- *continuous freq ω*
- *infinite x[n], -∞<n<∞*

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$



DFT – contd.

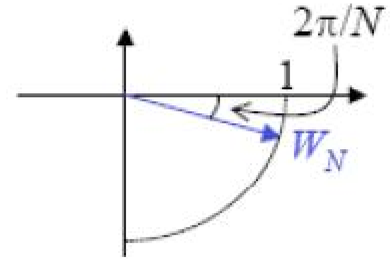
Hence,

$$\textbf{DFT:}\quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

- *discrete freq* $k = N\omega/2\pi$
- *finite* $x[n]$, $0 \le n < N$

Where, $W_N = e^{-j\frac{2\pi}{N}}$ *i.e, 1/N^{th} of a revolution*

Twiddle Factor

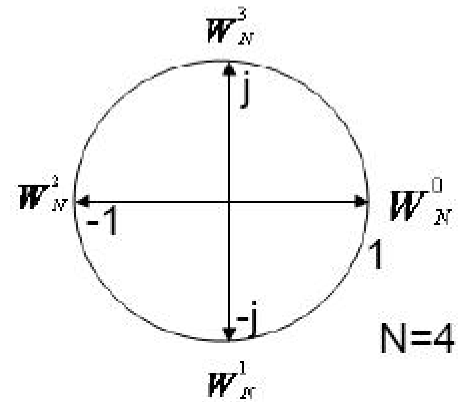


Properties of the Twiddle Factor:

$$W_N^{k+N} = W_N^{k} \longrightarrow \textit{periodicity}$$

$$W_N^{k+\frac{N}{2}} = -W_N^{k}$$

$$W_N^{2} = W_{\frac{N}{2}}$$



DFT Example

Find the DFT for the 4 points time sequence {1 0 0 1}, f_s=8KHz

- *at k=0,*

$$X(0) = \sum_{n=0}^{3} x(nT)\, e^{-j0} = \sum_{n=0}^{3} x(nT)$$

$$= x(0) + x(T) + x(2T) + x(3T)$$

$$= 1 + 0 + 0 + 1 = 2$$

- *at k=1,*

$$X(1) = \sum_{n=0}^{3} x(nT)\, e^{-j\Omega nT}, \Omega = {2\pi}/{NT}$$

$$X(1) = \sum_{n=0}^{3} x(nT)\, e^{-j2\pi n/N}$$

$$= 1 + 0 + 0 + 1e^{-j2\pi 3/4} = 1 + e^{-j3\pi/4}$$

$$= 1 + \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right) = 1 + j$$

DFT Example – contd.

- *at k=2,*

$$X(2) = \sum_{n=0}^{3} x(nT)\, e^{-j2\Omega nT} = \sum_{n=0}^{3} x(nT)\, e^{-j2\pi 2n/N}$$

$$= \sum_{n=0}^{3} x(nT)\, e^{-j4\pi n/N}$$

$$= 1 + 0 + 0 + 1e^{-j4\pi 3/4} = 1 + e^{-j3\pi} = 1 - 1 = 0$$

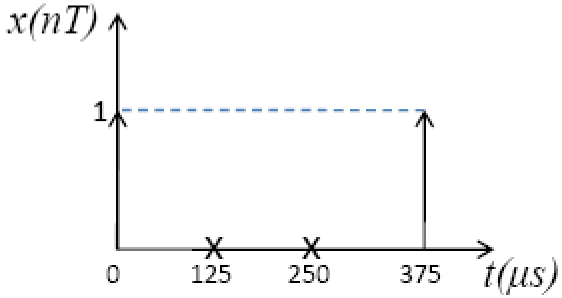
- *at k=3,*

$$X(3) = \sum_{n=0}^{3} x(nT)\, e^{-j2\pi 3n/N}$$

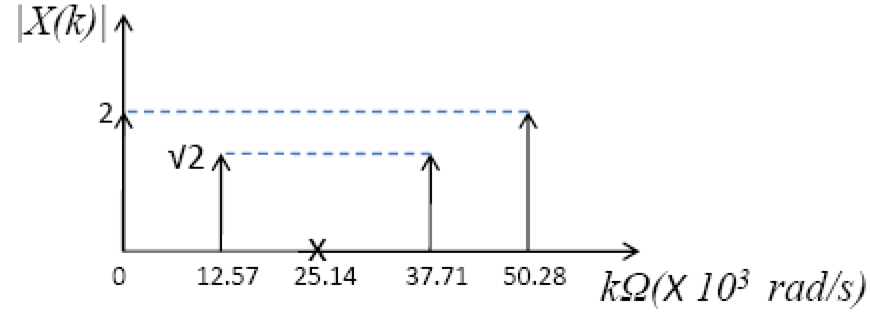
$$= 1 + 0 + 0 + 1e^{-j9\pi/2} = 1 - j$$

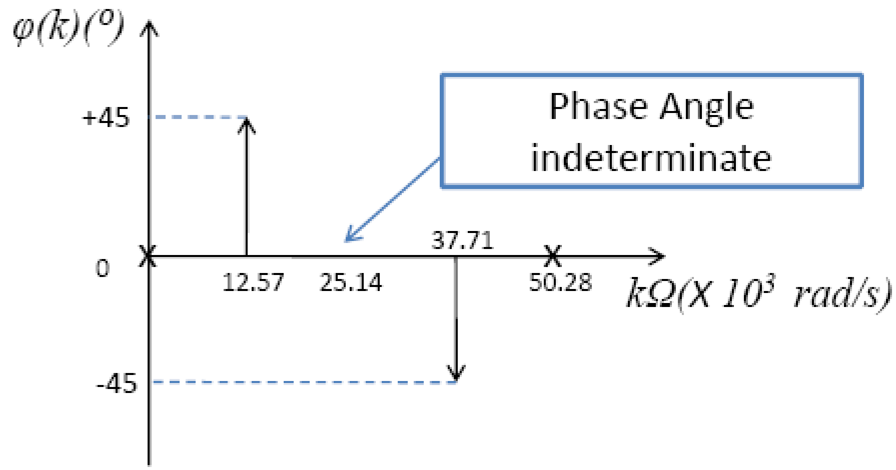
$$X(k) = \{2, 1 + j, 0, 1 - j\}$$

DFT Example – contd.



$$x(n) = \{1, 0, 0, 1\}$$





$$X(k) = \{2, 1 + j, 0, 1 - j\}$$

Inverse Discrete Fourier Transform (IDFT)

- **IDFT** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-nk}$

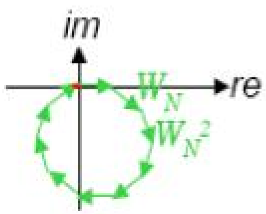
- *Check*

$$x[n] = \frac{1}{N}\sum_k \left(\sum_l x[l] W_N^{kl}\right) W_N^{-nk}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} W_N^{k(l-n)}$$

$$= x[n] \checkmark \quad 0 \le n < N$$

Sum of complete set of rotated vectors $= 0$ *if* $l \neq n$; $= N$ *if* $l = n$



DFT Computational Complexity

The DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

has:

- *(N complex multiplies + N-1 complex adds per point) x N points (k = 0..N-1)*

  N^2 *complex multiplies and N(N-1) complex additions*

where

cpx mult: (a+jb)(c+jd) = ac - bd + j(ad+bc)= 4 real mults + 2 real adds

cpx add = 2 real adds

- *Total:* $4N^2$ *real mults,* $4N^2-2N$ *real add*

- *Looking at DFT Matrix, lots of repeated structure are found; means opportunities for efficient algorithm to be used to reduce the DFT complexity*

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Lots of structure
→ opportunities for efficient algorithms

Fast Fourier Transform (FFT)

- *Reduce complexity of DFT from* $O(N^2)$ *to* $O(N \cdot \log_2 N)$
- *Grows more slowly with larger N*
- *Works by decomposing large DFT into several stages of smaller DFTs*
- *Often provided as a highly optimized library*

Decimation in Time (DIT) FFT

- *Can rearrange DFT formula in 2 halves:*

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}$$

$k = 0..\ N\text{-}1$

Arrange terms in pairs...

$$= \sum_{m=0}^{\frac{N}{2}-1} \left(x[2m] \cdot W_N^{2mk} + x[2m+1] \cdot W_N^{(2m+1)k} \right)$$

Group terms from each pair

$$= \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[2m] \cdot W_{\frac{N}{2}}^{mk}}_{X_0[\langle k \rangle_{N/2}]} + W_N^k \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[2m+1] \cdot W_{\frac{N}{2}}^{mk}}_{X_1[\langle k \rangle_{N/2}]}$$

$N/2$ pt DFT of x for **even** n $N/2$ pt DFT of x for **odd** n

Decimation in Time (DIT) FFT - contd.

$$DFT_N\{x[n]\} = DFT_{\frac{N}{2}}\{x_0[n]\} + W_N^K\, DFT_{\frac{N}{2}}\{x_1[n]\}$$

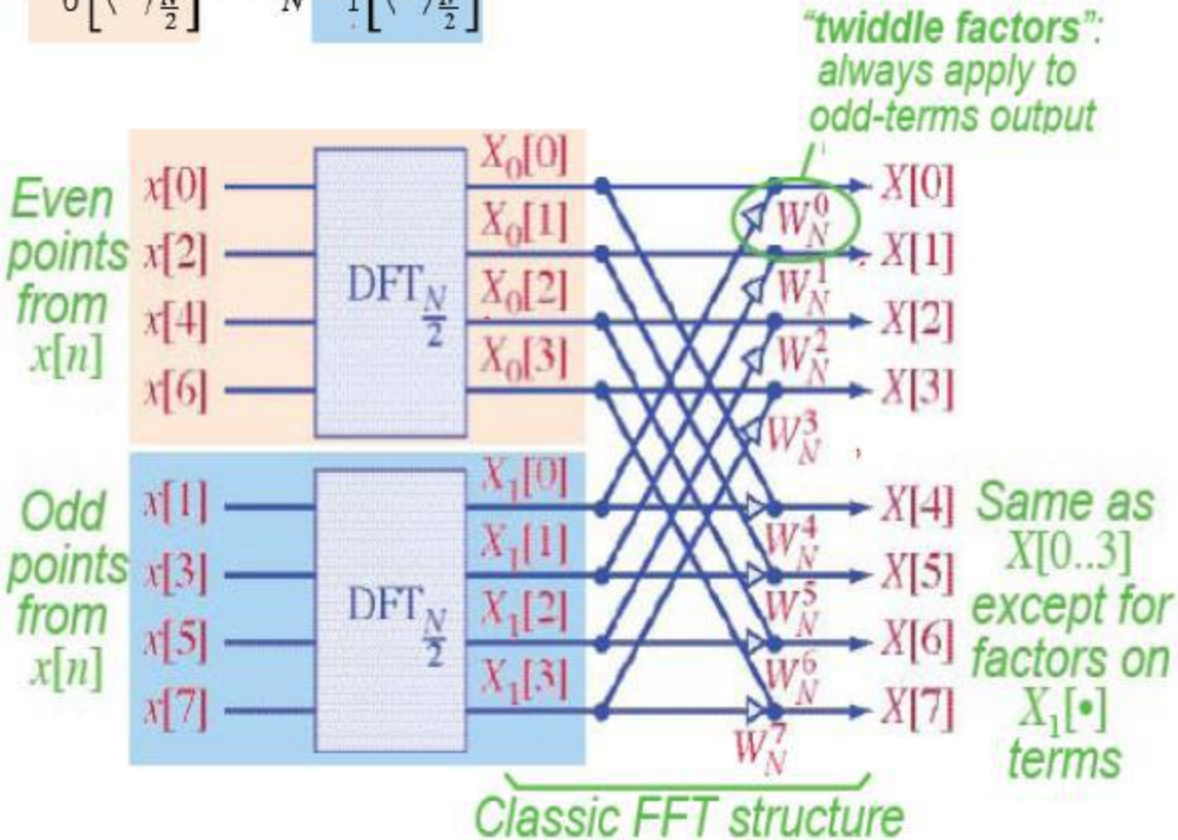
- *We can evaluate an N-pt DFT as two N/2-pt DFTs (plus a few mults/adds)*

- *But if* $DFT_N\{\bullet\} \sim O(N^2)$
  then $DFT_{N/2}\{\bullet\} \sim O((N/2)^2) = 1/4\, O(N^2)$

- *Total computation* $\sim 2 * 1/4\, O(N^2)$
  $= 1/2$ *the computation* $(+\varepsilon)$ *of direct DFT*

One-Stage DIT Flowgraph

$$X[k] = X_0\left[\langle k \rangle_{\frac{N}{2}}\right] + W_N^k X_1\left[\langle k \rangle_{\frac{N}{2}}\right]$$



Multiple DIT Stages

- *If decomposing one DFTN into two smaller $DFT_{N/2}$'s speeds things up...*
- *Why not further divide into $DFT_{N/4}$'s ?*
- *i.e.* $\underset{0 \le k < N}{X[k]} = X_0\left[\langle k \rangle_{\frac{N}{2}}\right] + W_N^k X_1\left[\langle k \rangle_{\frac{N}{2}}\right]$
- *so have* $\underset{0 \le k < N/2}{X_0[k]} = X_{00}\left[\langle k \rangle_{\frac{N}{4}}\right] + W_{\frac{N}{2}}^k X_{01}\left[\langle k \rangle_{\frac{N}{4}}\right]$

N/4-pt DFT of **even** points in **even** subset of $x[n]$

N/4-pt DFT of **odd** points from **even** subset

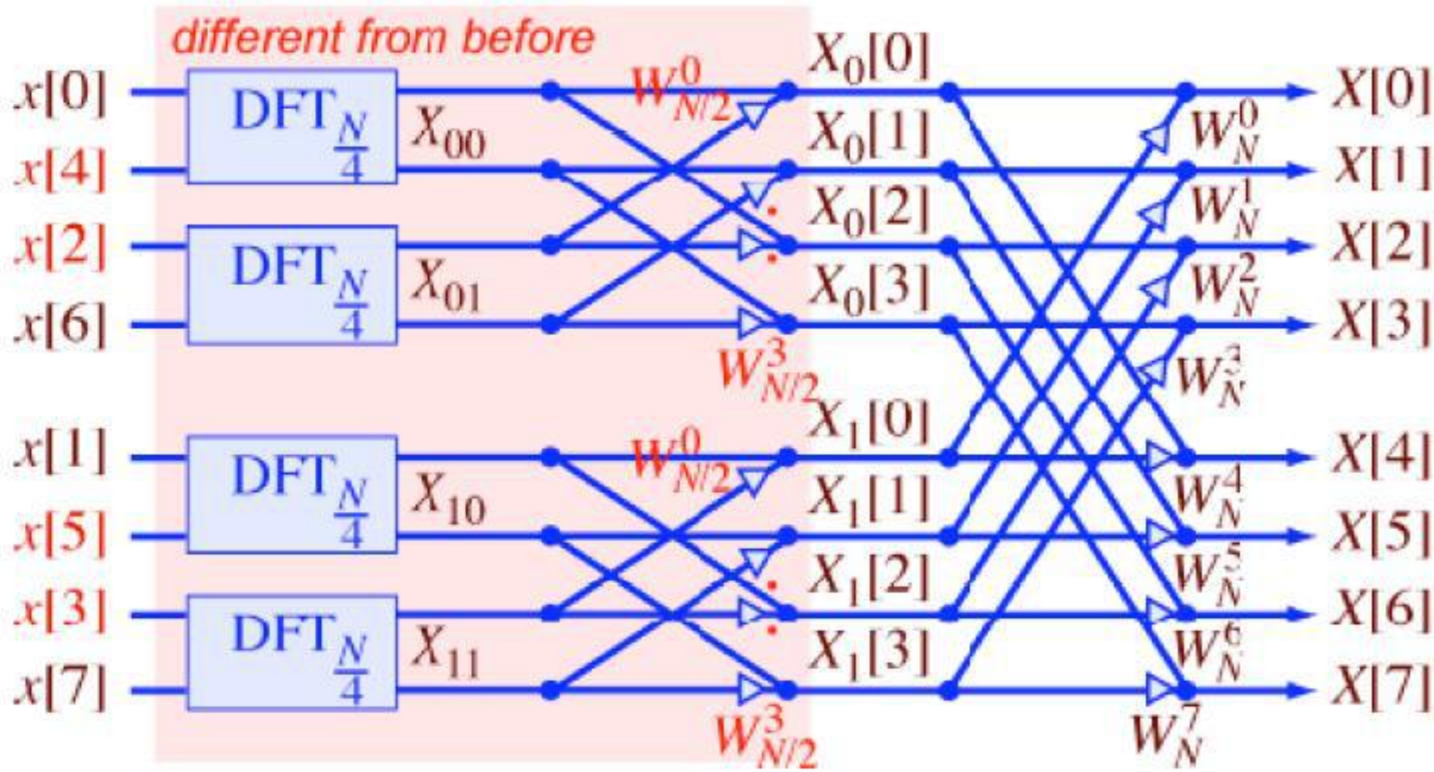
- *Similarly,* $X_1[k] = X_{10}\left[\langle k \rangle_{\frac{N}{4}}\right] + W_{\frac{N}{2}}^k X_{11}\left[\langle k \rangle_{\frac{N}{4}}\right]$

N/4-pt DFT of **even** points in **odd** subset of $x[n]$

N/4-pt DFT of **odd** points from **odd** subset

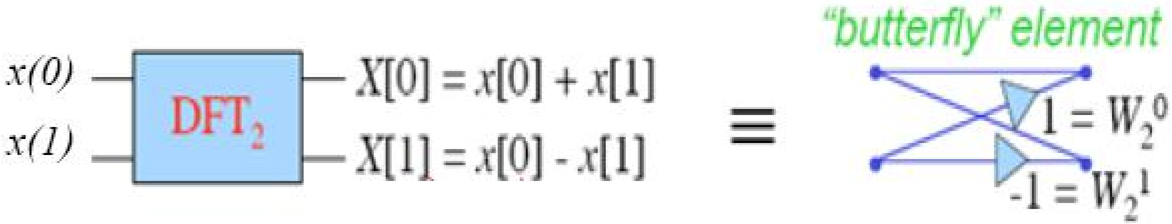
Multiple DIT Stages – contd.

Two-Stage DIT Flowgraph



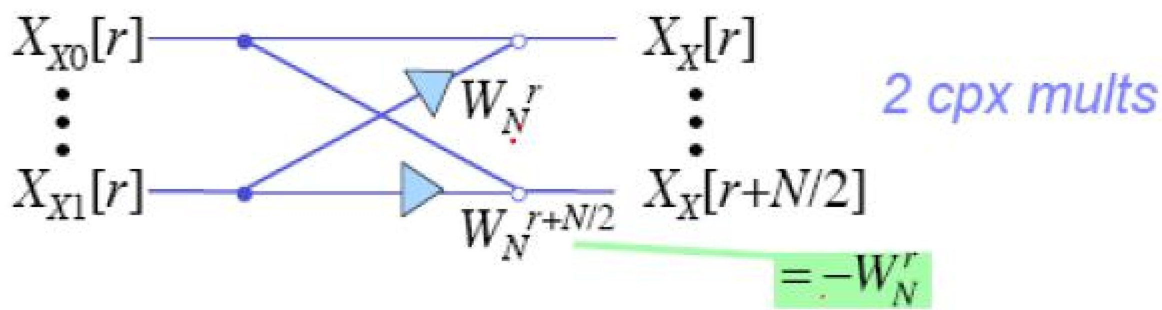
Multi-stage DIT FFT – contd.

- *Can keep doing this until we get down to 2-pt DFTs:*

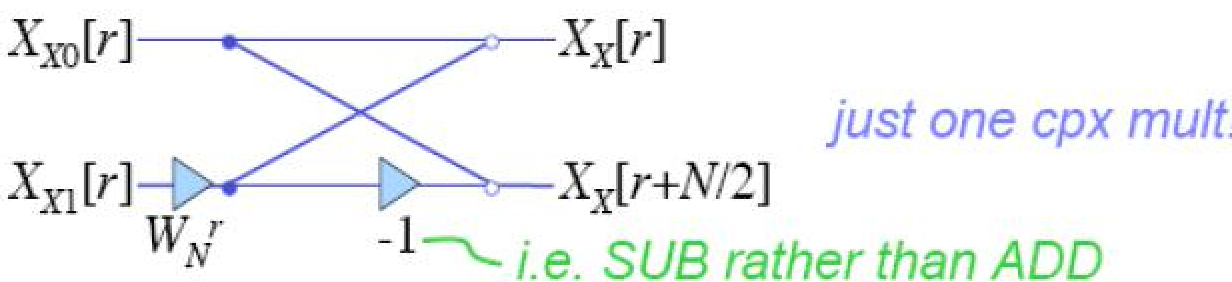


FFT Implementation Details

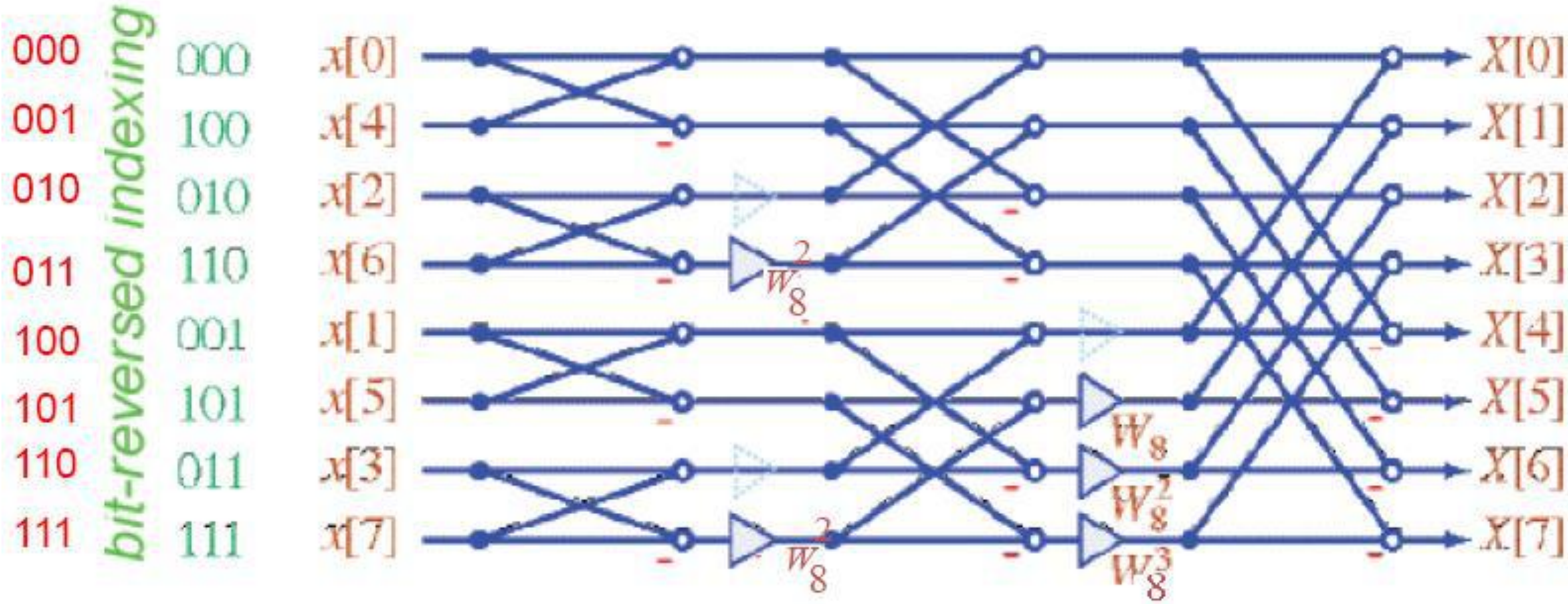
- *Basic Butterfly at any stage*



- *Can be simplified to*



8-pt DIT FFT Flowgraph



- *-1's absorbed into summation nodes*
- W_N^0 *disappears*
- *Twiddle factor step=N/2*^{j} *(stage order)*, $j \neq 1$

FFT Complexity

- *Total no. of butterflies =* $\frac{N}{2}\boxed{\log_2 N}$ → № of stages
- *As each butterfly gives one complex multiplication and 2 complex addition*
- *So, FFT has complex multiplications of*

$$O(\frac{N}{2}\log_2 N) \text{ instead of } O(N^2) \text{ in case of } DFT$$

and complex additions of

$$O(N\log_2 N) \text{ instead of } O(N(N-1)) \text{ in case of } DFT$$

Inverse FFT

only differences from forward DFT

- Recall IDFT: $x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-nk}$

- *Thus*

Forward DFT of $x'[n]=X^*[k]|_{k=n}$
i.e. time sequence made from spectrun

$$Nx^*[n]=\sum_{k=0}^{N-1}\left(X[k]W_N^{-nk}\right)^*=\sum_{k=0}^{N-1}X^*[k]W_N^{nk}$$

•***Hence, Use FFT to calculate IFFT***

$$x[n]=\tfrac{1}{N}\left[\sum_{k=0}^{N-1}X^*[k]W_N^{nk}\right]^*$$

Steps for calculating IFFT:

1. *Take the conjugate of the freq. Samples*
2. *Calculate FFT of these samples*
3. *Divide the output by N*
4. *Take the conjugate of the output*